MATEMATIKA I

$$\sum_{i=1}^{n} a_i := a_1 + a_2 + \ldots + a_n$$

$$\prod_{i=1}^{n} a_i := a_1 \cdot a_2 \cdot \ldots \cdot a_n$$

SUDOVI

⇒ oblik suda:

a: "5 > 3" ⇒ true (T, ⊤, 1)

b: "2 > 7" ⇒ false (F, ⊥, 0)

⇒ logičke operacije:

1. NEGACIJA ⇒ ¬ (čitamo: nije, non)
2. KONJUNKCIJA ⇒ ∧ (čitamo: i, et)
3. DISJUNKCIJA ⇒ ∨ (čitamo: ili, vel)
4. IMPLIKACIJA ⇒ ⇒ (čitamo: ima za posljedicu)
5. EKVIVALENCIJA ⇒ ⇔ (čitamo: jednakovrijedno je s)

x	$\neg x$
T	F
F	T

x	y	$x \wedge y$	$x \vee y$	$x \Rightarrow y$	$x \Leftrightarrow y$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

⇒ Boolova (logička) funkcija se iskazuje pomoću tablice istinitosti

PR: $\neg(X \wedge \neg Y)$

X	Y	$\neg Y$	$X \wedge \neg Y$	$\neg(X \wedge \neg Y)$
T	T	F	F	T
T	F	T	T	F
F	T	F	F	T
F	F	T	F	T

↳ vidimo: $\neg(X \wedge \neg Y) \equiv X \Rightarrow Y$

STAVAK 1:

a) $\neg(\neg X) \equiv X$

b) $X \wedge (Y \vee Z) \equiv (X \wedge Y) \vee (X \wedge Z)$
$X \vee (Y \wedge Z) \equiv (X \vee Y) \wedge (X \vee Z)$ } distributivnost

c) $\neg(X \wedge Y) \equiv \neg X \vee \neg Y$
$\neg(X \vee Y) \equiv \neg X \wedge \neg Y$ } De Morganove formule

d) $X \Leftrightarrow Y \equiv (X \Rightarrow Y) \wedge (Y \Rightarrow X)$

e) $X \Rightarrow Y \equiv \neg Y \Rightarrow \neg X$ } dokaz po kontrapoziciji

f) $\neg(X \Rightarrow Y) \equiv X \wedge \neg Y$

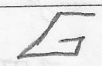
↳ Dokaz pod „c)":

X	Y	$X \wedge Y$	$\neg(X \wedge Y)$	$\neg X$	$\neg Y$	$\neg X \vee \neg Y$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

→ ekvivalentno ←

$\neg(X \wedge Y) \equiv \neg X \vee \neg Y$

PR: $\neg(\neg A \wedge (B \vee \neg C)) \equiv$ $\equiv A \vee (\neg B \wedge C)$



PR: $(X \Rightarrow Y) \vee (Y \Rightarrow X)$ $\Rightarrow$ TAUTOLOGIJA

PREDIKATI

$U \Rightarrow$ *univerzalni skup*

PR: $U = \mathbb{N}$

$P(x)$: „$x^2 < 5$"

$Q(x_1, x_2, x_3)$: „$x_1 + x_2 = x_3$"

↳ *tek znamo dal' su istina il' laž kada umjesto konkretne brojeve; to su „predikat" il „sudne forme"*

PR: $U = \mathbb{N}$

$P(x, y, z)$: „$x^2 + y^2 + z^2 < 3$" $\Rightarrow$ *uvijek lažno*

$\Rightarrow$ *„identički lažan predikat"*

$\Rightarrow$ KVANTIFIKATORI:

↳ oznake: $\forall$, $\exists$ → „za svak x je $P(x)$"

a) sud $\forall x\, P(x)$ je istinit, ako je $P(x) \equiv T$

(tj. $P(x)$ je identički istinit)

b) sud $\exists x\, P(x)$ je istinit, ako nije $P(x) \equiv F$

(tj. $P(x)$ nije identički lažan)

$\forall x\, P(x) := (P(x) \equiv T)$; $\exists x\, P(x) := \neg(P(x) \equiv F)$

STAVAK 3:

$P(x)$ je jednomjesni predikat

Onda vrijedi:

1. $\neg \forall x\, P(x) \equiv \exists x \neg P(x)$
2. $\neg \exists x\, P(x) \equiv \forall x \neg P(x)$
3. $\forall x\, P(x) \Rightarrow \exists x\, P(x)$

PRIRODNI BROJEVI I MATEMATIČKA INDUKCIJA

Koraci indukcije za: $A(n)$, $n \in N$

1. $A(n_0) = T$; $n_0 \in T$ → baza
2. $A(n) = T \Longrightarrow A(n+1) = T$; $n > n_0$ → korak
3. $A(n) = T$ $\forall n \in N$, $n > n_0$ → zaključak

↳ pogledaj PR. 28 ; PR. 29

ZAD: $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \quad \forall n \in N$

① $n = 1 \qquad 1 = 1 \qquad T$

② $\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \Rightarrow \sum_{i=1}^{n+1} i \cdot i! \overset{(?)}{=} (n+2)! - 1$

$$\sum_{i=1}^{n+1} i \cdot i! = \sum_{i=1}^{n} i \cdot i! + (n+1)(n+1)!$$

$$= (n+1)! - 1 + (n+1)(n+1)!$$
$$= ((n+1)!)(1 + n + 1) - 1$$
$$= (n+1)!\,(n+2) - 1$$
$$= (n+2)! - 1 \quad \checkmark$$

③ $\left(\sum_{i=1}^{n} i \cdot i! = (n+1)! - 1 \right) \forall n \in N$

ZAD: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$; $\forall n \in N$ ☐

ZAD: $5^n + 2^{n-1}$ djeljivo s 3, $\forall n \in N$

① $n=1$ $\quad 5^1 + 2^0 = 6$ T

② $5^n + 2^{n-1}$ djeljivo s 3

↳ $5^{n+1} + 2^n = 5 \cdot 5^n + 2 \cdot 2^{n-1}$

$= 3 \cdot 5^n + 2 \cdot 5^n + 2 \cdot 2^{n-1}$

$= \underbrace{3 \cdot 5^n}_{\text{djeljivo s 3}} + 2 \underbrace{(5^n + 2^{n-1})}_{\text{djeljivo s 3}}$ ✓

③ $5^n + 2^{n-1}$ djeljivo s 3, $\forall n \in N$

ZAD: $3^n (n!)^2 < (2n)!$; $\forall n \in N$, $n \geq 5$ ☐

① $n=1$ $\quad 3 \cdot 1^2 < 2$ N

$n=5$ $\quad 3^5 (5!)^2 < (2 \cdot 5)!$ T

② $3^n (n!)^2 < (2n)! \Rightarrow 3^{n+1} ((n+1)!)^2 < (2(n+1))!$

$3^{n+1} (n+1)! \cdot (n+1)! < (2n+2)!$

$3 \cdot 3^n (n+1)^2 (n!)^2 < (2n)! (2n+1)(2n+2)$

↓ treba pokazati:

$3(n+1)^2 < (2n+2)(2n+1)$

$3n^2 + 6n + 3 < 4n^2 + 6n + 2$

$n^2 > 1$ ✓ (vrijedi jer je $n \geq 5$)

③ $3^n (n!)^2 < (2n)!$; $\forall n \in N$

⇒ BINO

BINOMNI KOEFICIENT

$n \in \mathbb{N}$; $k \in \{0, 1, 2, 3, 4, \ldots, n\}$

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k-2)(n-k-1)}{k!} = \frac{n(n-1)\cdots(n-k+1)}{k!} \cdot \frac{(n-k)(n-k-1)\cdots 1}{(n-k)(n-k-1)\cdots}$$

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

⇒ vrijedi simetrija: $\binom{n}{k} = \binom{n}{n-k}$

PR: Dokaži: $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ ☐ PR: 30

⇒ BINOMNA FORMULA: $a, b \in \mathbb{R}$; $n \in \mathbb{N}$

$$(a+b)^n = a + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \ldots + \binom{n}{n-1} a b^{n-1} + b^n$$

„ili"

$$(a+b)^2 = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

KOMPLEKSNI BROJEVI

$C = \{ a + bi : a, b \in R ;\ i^2 = -1 \}$ //

⇒ *lakše shvaćanje skupa C:*

$C = \{(a, b) : a, b \in R\} = R \times R$

$(a, b) = (c, d) \iff a = c \wedge b = d$

$(a, b), (c, d) \in C$

$(a, b) \oplus (c, d) := (a + c, b + d) \in C$

↳ neko „čudno" zbrajanje //

$(a, b) \odot (c, d) := (ac - bd, ad + bc)$

↳ neko „čudno" množenje //

$\frac{(a, b)}{(c, d)} = (e, f) \Rightarrow (c, d) \cdot (e, f) = (a, b)\ ;\ (c, d) \neq (0, 0)$

$c \neq 0 \vee d \neq 0$ //

↳ neka pravila u C: $\boxed{(x, 0) = x}$

$\underbrace{(0, 1)}_{i} \cdot (0, 1) = (0 \cdot 0 - 1 \cdot 1,\ 0 \cdot 1 + 1 \cdot 1)$

$= (-1, 0) = -1$

⇓

$\boxed{i := (0, 1)}$ //

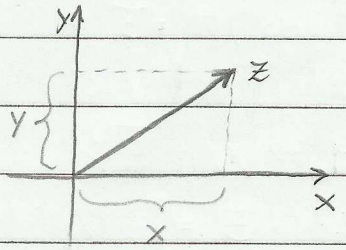
$(a, b) = (a, 0) + (0, b)$

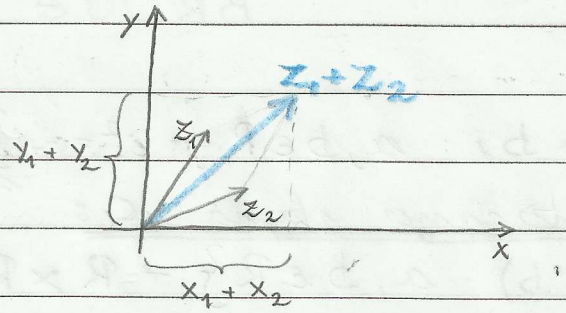
$= a + \underbrace{(0, 1)}_{i}\underbrace{(b, 0)}_{b} = a + bi$ //

⇒ PROBAJ: $(a + ib) \cdot (c + id) = \ldots = (ac - bd) + (ad + bc)i$

↳ neko „čudno" množenje je zapravo isto kao i „normalno" množenje ! Probaj !

⇒ prikaz u Kartezijevoj ravnini: $z = x + yi$

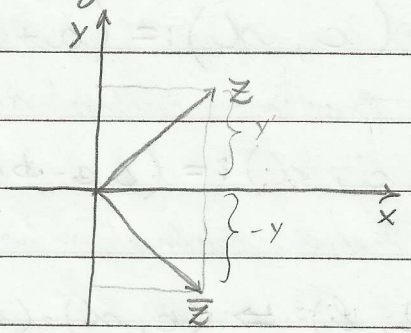




⇒ konjugirano kompleksni broj:

$z = x + yi$

$\bar{z} = x - yi$



↳ pravila: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

$\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

$\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$

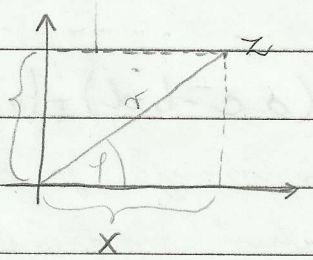
$|z| = \sqrt{x^2 + y^2}$ → zbroj kvadrata

$z \cdot \bar{z} = (x + yi)(x - yi) = x^2 + y^2 = |z|^2$

$|z_1 \cdot z_2| = \sqrt{(z_1 \cdot z_2)\overline{(z_1 z_2)}} = |z_1||z_2|$

⇒ trigonometrijski oblik kompleksnog broja:

$z = x + y \qquad T(x, y)$



$|z| = \sqrt{x^2 + y^2}$

$r = \sqrt{x^2 + y^2}$ → modul

$\tan \varphi = \frac{y}{x}$ → argument

razne jednadžbe

$$\boxed{\begin{aligned} &\operatorname{Arg} z = \varphi + k \cdot 2\pi \;;\; k \in \mathbb{Z} \\ &|z| = r \end{aligned}}$$

$$x = r\cos\varphi \;;\; y = r\sin\varphi$$

$$z = x + yi = r\cos\varphi + ir\sin\varphi$$

$$\boxed{z = r\cdot(\cos\varphi + i\sin\varphi)}$$

PR: $z_1 = r_1(\cos\varphi_1 + i\sin\varphi_1)$

$z_2 = r_2(\cos\varphi_2 + i\sin\varphi_2)$

$z_1 \cdot z_2 = ?$

$$z_1\cdot z_2 = r_1\cdot r_2\left(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2 + i\left(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2\right)\right)$$

$$z_1\cdot z_2 = r_1\cdot r_2\left(\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)\right)$$

$$|z_1\cdot z_2| = r_1\cdot r_2 = |z_1|\cdot|z_2|$$

$$\arg(z_1\cdot z_2) = \arg z_1 + \arg z_2$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left(\cos(\varphi_1-\varphi_2) + i\sin(\varphi_1-\varphi_2)\right)$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} = \frac{r_1}{r_2} \qquad \arg\frac{z_1}{z_2} = \arg z_1 - \arg z_2$$

$$z^n = r^n(\cos n\varphi + i\sin n\varphi)$$

⇒ eksponencijalni zapis kompleksnog broja:

$$e^{i\varphi} := \cos\varphi + i\sin\varphi$$

$$\boxed{z = r\cdot e^{i\varphi}}$$

$$\boxed{\begin{aligned} &z_1\cdot z_2 = r_1\cdot r_2\cdot e^{i(\varphi_1+\varphi_2)} \\ &\frac{z_1}{z_2} = \frac{r_1}{r_2}\cdot e^{i(\varphi_1-\varphi_2)} \\ &z^n = r^n\cdot e^{in\varphi} \end{aligned}}$$

⇒ korijen kompleksnog broja:

$$z = r(\cos\varphi + i\sin\varphi)$$

$$\sqrt[n]{z} = w = \rho(\cos\psi + i\sin\psi)$$

$$w^n = z$$

$$\rho^n(\cos n\psi + i\sin n\psi) = r(\cos\varphi + i\sin\varphi)$$

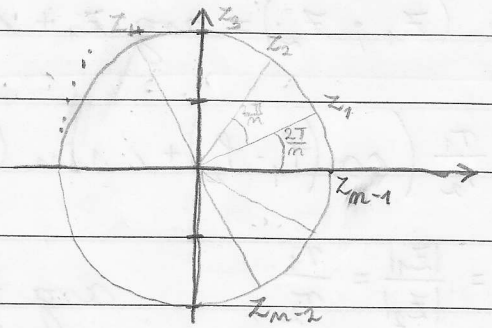
↳ vrijedi kada: $\rho^n = r \Rightarrow \rho = \sqrt[n]{r}$

$$n\cdot\psi = \varphi + 2k\pi \Rightarrow \psi = \frac{\varphi + 2k\pi}{n}$$

$k = 0, 1, 2, \ldots, n-1$

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\frac{\varphi+2k\pi}{n} + i\sin\frac{\varphi+2k\pi}{n}\right)$$

↳ kompleksni su brojevi na kružnici razmaknuti za $\frac{2\pi}{n}$ radijana



SKUPOVI I PRESLIKAVANJA

$S = \{x \in N : x < 4\} = \{1, 2, 3\} = \{x \in N : x^2 < 12\}$

⇒ OPERACIJE NAD SKUPOVIMA:

a) $\overline{S} = \{x : x \notin S\}$ ⇒ komplement

b) $S_1 \cup S_2 = \{x : x \in S_1 \vee x \in S_2\}$ ⇒ unija

c) $S_1 \cap S_2 = \{x : x \in S_1 \wedge x \in S_2\}$ ⇒ presjek

d) $S_1 \setminus S_2 = \{x : x \in S_1 \wedge x \notin S_2\}$ ⇒ razlika

⇒ KARTEZIJEV UMNOŽAK:

$S_1 \times S_2 = \{(x, y) : x \in S_1 \wedge y \in S_2\}$

↳ uređeni parovi

PR: $S_1 = \{1, 2\}$ $S_2 = \{3, 4\}$

$S_1 \times S_2 = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$

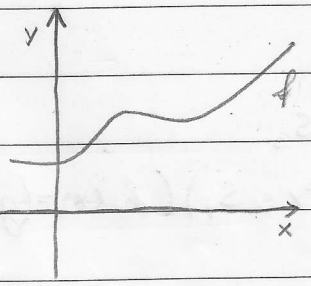
⇒ PRESLIKAVANJA:

↳ DEF: $f : S_1 \to S_2$ je preslikavanje iz S_1 u S_2 ako je svakom $x \in S_1$ pridružen jedinstven $x \in S_2$.

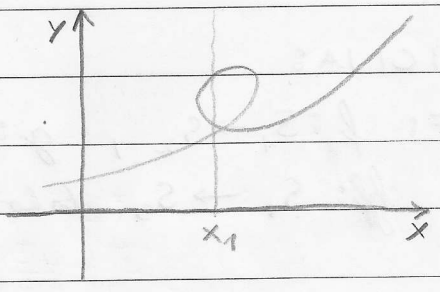
$y = f(x)$ ⇒ funkcija

$D(f) = S_1$ ⇒ domena, područje definicije

$Im(f) = S_2$ ⇒ slika funkcije



↳ funkcija



↳ nije funkcija

⇒ INJEKCIJA, SURJEKCIJA, BIJEKCIJA:

$$f: S_1 \to S_2, \quad D(f) = S_1, \quad f(S_1) \subseteq S_2$$

a) f je surjekcija ako $f(S_1) = S_2$

↳ to jest: $(\forall y \in S_2)(\exists x \in S_1)(f(x) = y)$

b) f je injekcija ako se „različiti preslikavaju u različite"

↳ to jest: $(\forall x_1, x_2 \in S_1)(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2))$

c) f je bijekcija ako je i injekcija i surjekcija

PR: a) Što znači da $f: S_1 \to S_2$ nije surjekcija?

↳ negiraj definiciju:

$$\neg((\forall y \in S_2)(\exists x \in S_1)(f(x) = y))$$
$$\equiv (\exists y \in S_2)(\forall x \in S_1)(f(x) \neq y)$$

↳ riječima: postoji neki y iz S_2 koji nije slika niti jednog x iz S_1

b) Što znači da $f: S_1 \to S_2$ nije injekcija?

↳ negiraj definiciju

$$\neg((\forall x_1, x_2 \in S_1)(x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)))$$
$$\equiv (\exists x_1, x_2 \in S_1)(x_1 \neq x_2 \wedge f(x_1) = f(x_2))$$

⇒ KOMPOZICIJA:

↳ DEF: $f: S_1 \to S_2$, $g: S_2 \to S_3$

$h: S_1 \to S_3$ tako da $(\forall x \in S_1)(h(x) = (g \circ f)(x) = g(f(x)))$

PR: $h_1(x) = (\sqrt{x})^2$, $D(h_1) = R_0^+$ $\quad$ $h_2(x) = \sqrt{x^2}$, $D(h_2) = R$ $\quad\Big\}$ $h_1 \neq h_2$ jer su domene različite

⇒ INVERZNA FUNKCIJA:

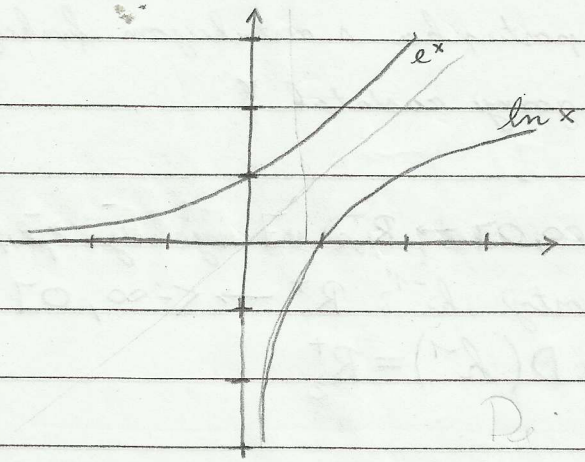
↳ DEF: $f: S_1 \to S_2$

↳ preslikavanje $g: S_2 \to S_1$ se zove inverzno preslikavanje od f ako:

$$(\forall x \in S_1)(g(f(x)) = x) \quad i \quad (\forall y \in S_2)(f(g(y)) = y)$$

↳ pišemo: $g = f^{-1}$

PR: $f(x) = e^x$, $f: R \to R^+$; nađi inverznu!



Tražimo: $f^{-1}: R^+ \to R$

$f(f^{-1}(x)) = x$; $\forall x \in D(f^{-1}) = R^+$

$e^{f^{-1}(x)} = x$, $\forall x \in R^+$

$f^{-1}(x) = \ln x$

⇒ FORMALNO TRAŽENJE INVERZNE FUNKCIJE:

$y = f(x) = e^x$

↳ "zamijenimo" x i y ⇒ $x = e^y$

↳ "izrazimo" y ⇒ $y = \ln x = f^{-1}(x)$ //

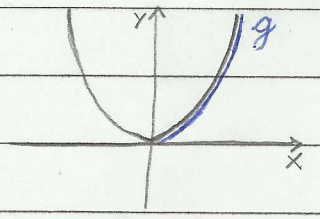
PR: Restrikcija funkcije;

$f(x) = x^2$

$D(f) = R$; $Im(f) = R^+$

↳ f nije injekcija!

↳ stoga, uzmimo restrikciju na $[0, +\infty\rangle$



$f|_{[0,+\infty\rangle} \equiv g$; $g: [0, +\infty\rangle \to R_0^+$ je bijekcija, pa $\exists g^{-1}: R_0^+ \to [0, +\infty\rangle$

OKRENI

$g(g^{-1}(x)) = x \quad , \quad \forall x \in D(g^{-1}) = R^+$

$(g^{-1}(x))^2 = x \; ; \quad \forall x \geq 0$ pa možemo korjenovat

$g^{-1}(x) = \sqrt{x}$

↳ formalan način: $Y = X^2$

$X \in [0, +\infty\rangle \; ; \; Y \in [0, +\infty\rangle$

$X = Y^2$

$Y = \sqrt{X}$

⇒napomena: pazi kod svakog postupka s restrikcijom funkcija

⇒ DZ: $h \equiv f|_{\langle -\infty, 0]}$ i $h^{-1} = ?$ za gornji zadatak ☺

↳ $f|_{\langle -\infty, 0]} \equiv h \; ; \; „h": \langle -\infty, 0] \to R_0^+$ je injekcija, pa postoj $„h^{-1}": R_0^+ \to \langle -\infty, 0]$

$h(h^{-1}(x)) = x \quad \forall x \in D(h^{-1}) = R_0^+$

$(h^{-1}(x))^2 = x$

$h^{-1}(x) = \sqrt{x}$

PRIMJER BIJEKCIJE: Identitet: $S \to S$

$f(x) = x$

FUNKCIJE

⇒ bavimo se $f: \mathbb{R} \to \mathbb{R}$, $y = f(x)$

$D(f)$ ⇒ područje definicije

$Im(f) = \{y \in \mathbb{R} : y = f(x), x \in D(f)\}$

↳ GRAF FUNKCIJE: $G(f) = \{(x, y) \in \mathbb{R}^2 : x \in D(f), y = f(x)\}$

ZAD: Odredi domenu: $f(x) = \sqrt{x^3 - 3x^2 - 10x + 24}$

$x^3 - 3x^2 - 10x + 24 \geq 0$

$(x^3 - 3x^2 - 10x + 24) : (x - 2) = x^2 - x - 12$

$-(x^3 - 2x^2)$

$-x^2 - 10x + 24$

$-(-x^2 + 2x)$

$-12x + 24$

$-(-12x + 24)$

0

faktorizirano: $(x-4)(x+3)$

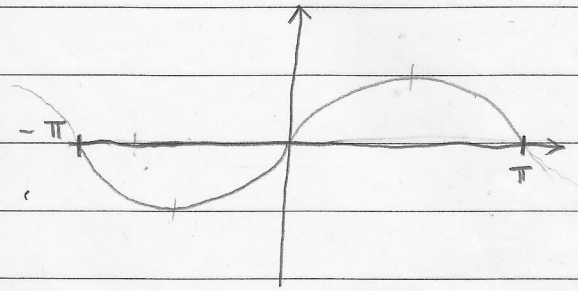
$x^3 - 3x^2 - 10x + 24 = (x-2)(x+3)(x-4) \geq 0$

	$\langle -\infty, -3 \rangle$	$\langle -3, 2 \rangle$	$\langle 2, 4 \rangle$	$\langle 4, +\infty \rangle$
$x-2$	$-$	$-$	0 $+$	$+$
$x-4$	$-$	$-$	$-$	0 $+$
$x+3$	$-$	0 $+$	$+$	$+$

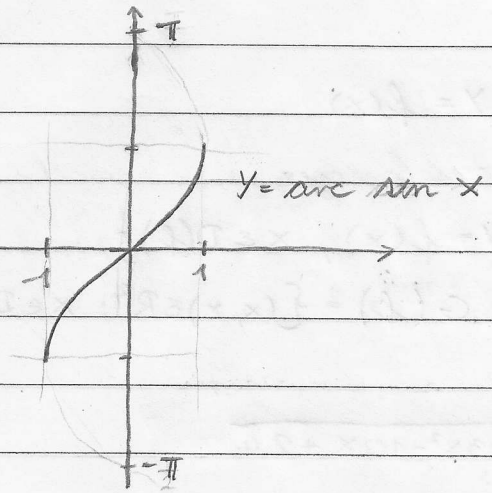
PR: $f(x) = \sin x$ → je li funkcija injektivna?

↳ nije, ali na intervalu:

$\sin: [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$ je funkcija bijektivna



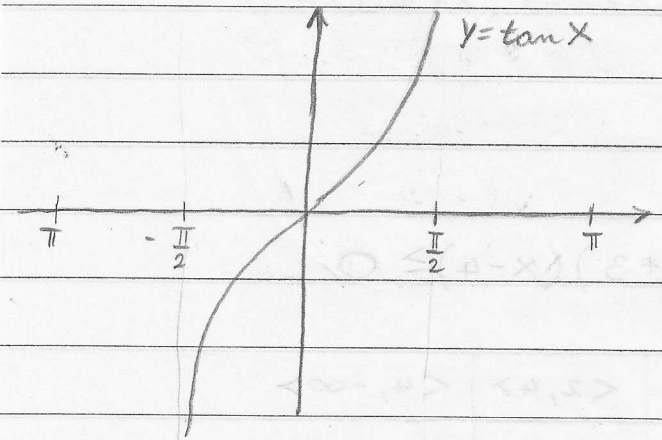
PR: $f(x) = \arcsin x : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ je bijekcija



npr. $\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2} \Leftrightarrow \arcsin \frac{\sqrt{3}}{2} = \frac{\pi}{3}$

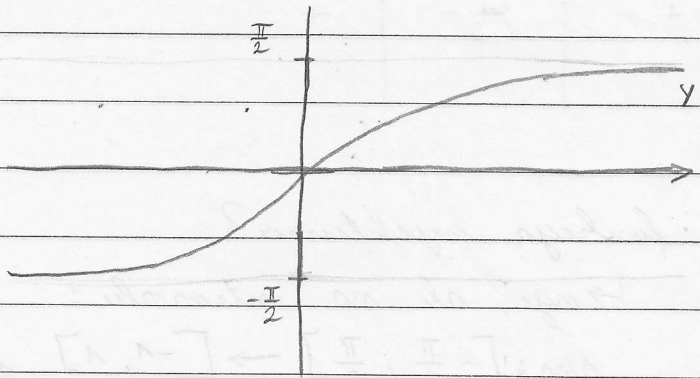
PR: $f(x) = \tan x = \frac{\sin x}{\cos x}$

$tg : (-\frac{\pi}{2}, \frac{\pi}{2}) \to R$ je bijekcija



PR: $f(x) = \arctan x$

$y = \tan x \Leftrightarrow x = \arctan y$



HIPERBOLIČKE FUNKCIJE

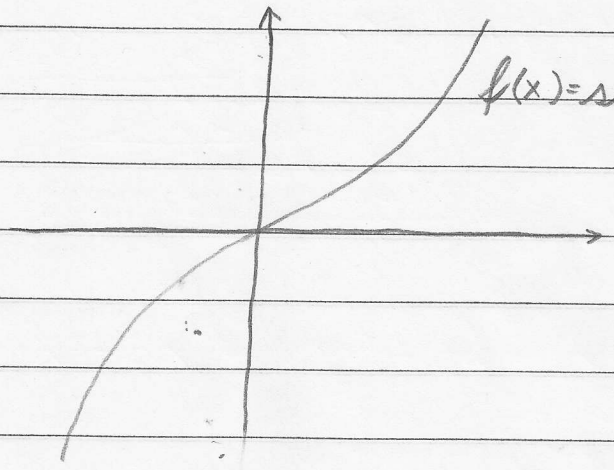
PR: 1) $\boxed{sh\,x \overset{def}{=} \frac{e^x - e^{-x}}{2}}$ ⇒ to je „sinus-hiperbolički od x"

$$sh(-x) = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} = -sh(x)$$

↳ neparna funkcija

↳ INVERZNO: $y = sh\,x \Leftrightarrow x = ar\,sh\,y$

„area..."



$sh : R \to R$

↳ bijekcija

$ar\,sh : R \to R$

↳ bijekcija

2) Izračunajmo: „$ar\,sh\,x$" = y

$$y = ar\,sh\,x \Leftrightarrow x = sh\,y$$

$$x = \frac{e^y - e^{-y}}{2}$$

$$e^y - e^{-y} = 2x \quad / \cdot e^y$$

$$e^{2y} - 2x e^y - 1 = 0$$

$u = e^y > 0$!

$$u^2 - 2xu - 1 = 0$$

$$u_{1,2} = \frac{2x \pm \sqrt{4x^2+4}}{2}$$ („minus" otpada jer je $u > 0$)

$$u = x + \sqrt{x^2+1} = e^y \quad / \ln$$

$$\boxed{y = \ln\left(x + \sqrt{x^2+1}\right)}$$

DEF: funkcija $f: R \to R$ je <u>neparna</u>, ako je $f(-x) = -f(x)$ $\forall x \in D(f)$

↳ simetrija na ishodište

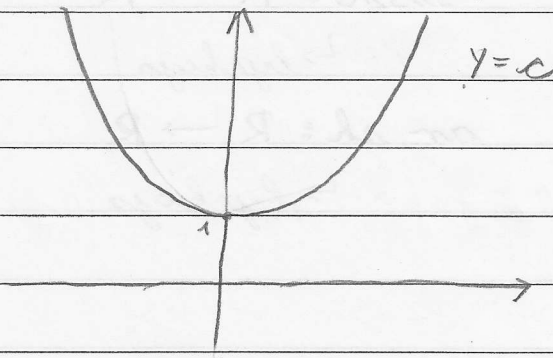
DEF: funkcija $f: R \to R$ je <u>parna</u>, ako je $f(-x) = f(x)$ $\forall x \in D(f)$

↳ simetrija na „y-os"

PR: 1) $y = \boxed{ch\, x = \frac{e^x + e^{-x}}{2}}$ ⇒ to je „kosinus - hiperbolički od x".

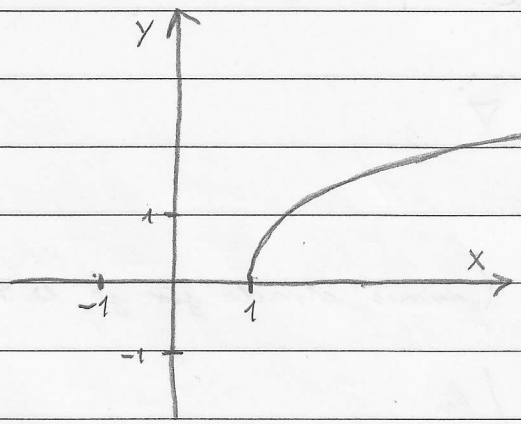
$ch(-x) = ch\, x$

↳ parna funkcija



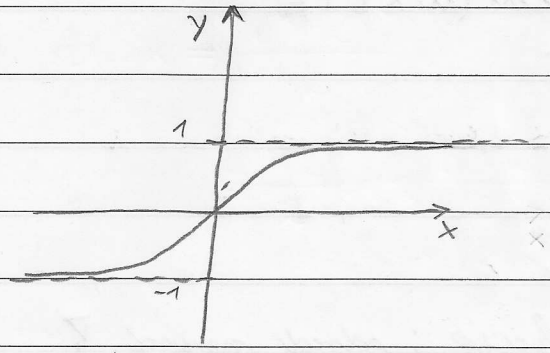
2) $y = ch\, x \iff x = ar\, ch\, y$ (zamjenimo x, y : $y = ar\, ch\, x$)

↳ $y = ch\, x$ nije bijekcija, pa smijemo restrikciju na interval $[0, \infty\rangle$

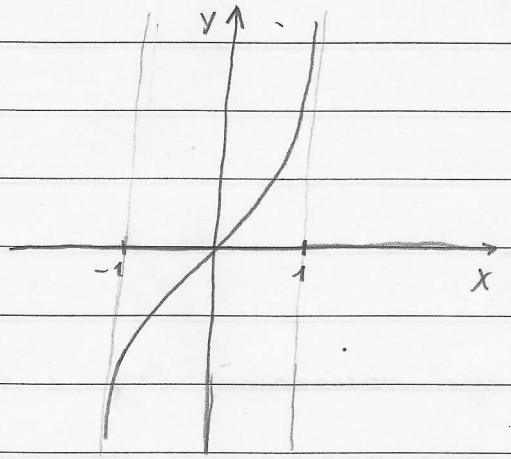


PR: 1) $th\,x = \frac{sh\,x}{ch\,x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

$th(-x) = -th\,x \Rightarrow$ neparna funkcija



2) $y = ar\,th\,x \Leftrightarrow x = th\,y$



DEF: funkcija $f: R \to R$ je periodička ako postoj $T > 0$ tako da vrijed: $f(x+T) = f(x)$, $\forall x \in D(f)$

↳ pretpostavka: $x \in D(f) \Rightarrow (x+T) \in D(f)$

↳ minimalni period je najmanji $T > 0$ (TEMELJNI PERIOD)

↳ napomena: konstante jesu periodičke, ali nemaju temeljni period

PR: $y = A\sin(\omega x + \varphi)$; $x \in R$

$(A > 0;\ \omega > 0;\ \varphi \in R)$ su konstante

↳ izrazi njen period

OKRENI

$$f(x+T) = A\sin(\omega(x+T)+\varphi) \underset{\text{pretpostavka}}{=} f(x) = A\sin(\omega x+\varphi)$$

$$\sin(\omega x+\varphi+\omega T) = \sin(\omega x+\varphi)$$

$$\omega T = 2\pi$$

$$\boxed{T = \frac{2\pi}{\omega}}$$

PR: Zbroj dvije periodičke funkcije; odredi period!

a) $Y = \underbrace{\sin 5x}_{\frac{2\pi}{5}} + \underbrace{\cos 3x}_{\frac{2\pi}{3}}$

$$k\cdot\frac{2\pi}{5} = l\cdot\frac{2\pi}{3}$$

$$3k = 5l$$

$$k=5;\ l=3$$

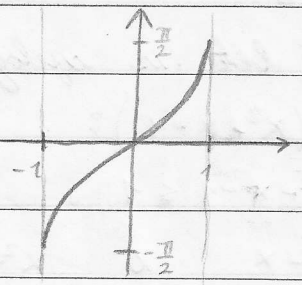
b) $Y = \sin x + \sin(\sqrt{2}\,x)$

↳ nije periodička jer $\sqrt{2}$ nije racionalan broj i nema zajedničkih višekratnika

DOMENE FUNKCIJA

PR: $y = \arcsin x$

$D(f) = [-1, 1]$



PR: Odredi domenu: $f(x) = \arcsin(3x-2)$

$3x-2 \in [-1, 1]$

$-1 \leq 3x-2 \leq 1$

$\frac{1}{2} \leq x \leq 1$

$D(f) = [\frac{1}{3}, 1]$

PR: Odredi domenu funkcije: $f(x) = \ln(x^2-4) \cdot \frac{\sqrt{\ln x - 1}}{ch\, x}$

a) gledaj $\ln(x^2-4)$:

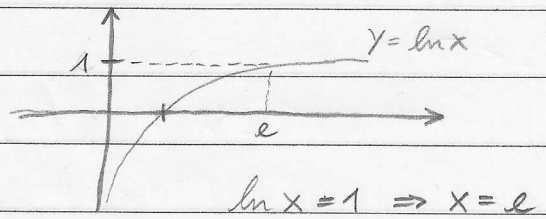
$x^2 - 4 > 0$

$x^2 > 4 \Rightarrow x \in \langle -\infty, -2] \cup \langle 2, \infty \rangle$

b) gledaj $\sqrt{\ln x - 1}$

$\ln x \geq 1$

$x \geq e$



$\ln x = 1 \Rightarrow x = e$

c) gledaj $ch\, x$:

$ch\, x \geq 1$ tako da „nema problema"

↳ sada treba naći presjek rješenja pod a) i b):

$x \in [e, \infty \rangle$

$D(f) = [e, \infty \rangle$

PR: Odredi domenu funkcije: $f(x) = (-x+4)^{1/3} + \arcsin(x^2-x-1)$

a) gledaj $(-x+4)^{1/3}$:

↳ x može biti iz cijelog R

b) gledaj $\arcsin(x^2-x-1)$:

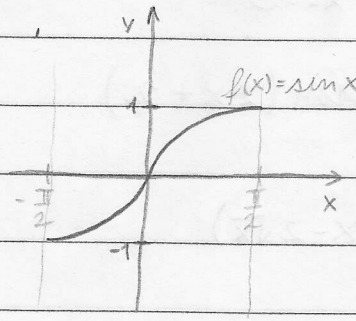
$-1 \leq x^2-x-1 \leq 1$

$0 \leq x^2-x$ i $x^2-x-2 \leq 0$

$x \in \langle -\infty, 0] \cup [1, \infty \rangle$ i $x \in$

ARKUS FUNKCIJE

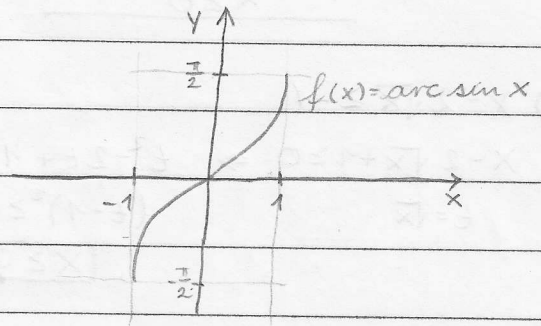
PR: $f(x) = \sin x \qquad R \to [-1, 1]$



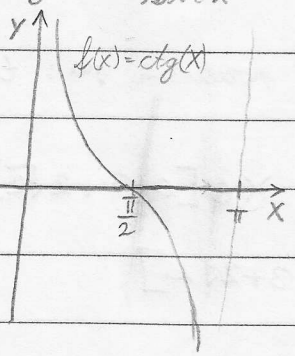
$f|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} \equiv Sin$

$Sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1]$

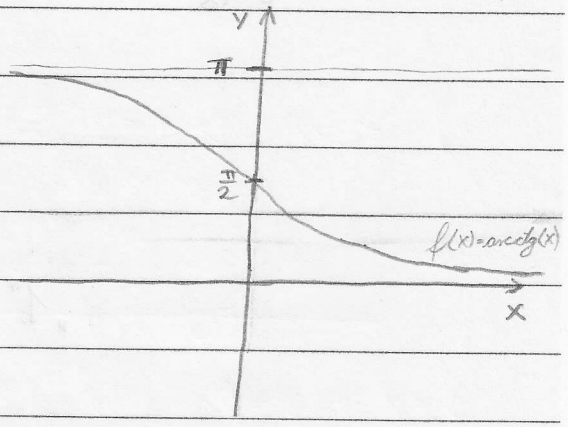
$arc\,sin : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$



PR: $f(x) = ctg\,x = \frac{\cos x}{\sin x} : R \setminus \bigcup_{k \in Z} \{k\pi\} \to R$



$f|_{[0, \pi]} \equiv Ctg$



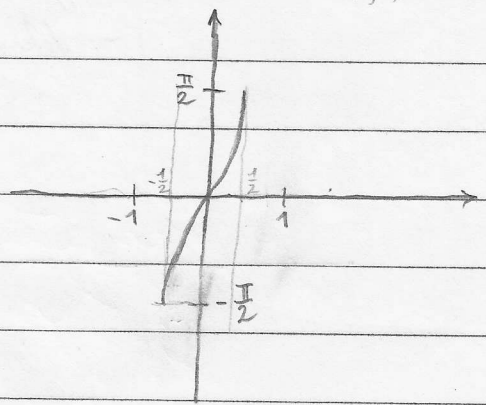
ZAD: Skiciraj $f(x) = arc\,sin(2x)$ i odredi $D(f)$ ∇:

$-1 \le 2x \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$

$D_f = [-\frac{1}{2}, \frac{1}{2}]$

$Im\,f = [-\frac{\pi}{2}, \frac{\pi}{2}]$



ZAD: Odvod jednadžbu translatiranog grafa funkcije $f(x) = \arcsin(2x)$ za 1 u lijevo!

$$\widetilde{f}(x) = f(x+c)$$

pomicanje nultočke ↳ najjednostavnije!

$$x + c = 0$$
$$x = -c$$

$$\widetilde{f}(x) = \arcsin(2(x+1)) = \arcsin(2x+2)$$ ☐

ZAD: Odredi $D(g)$, ako je $g(x) = \arcsin(x - 2\sqrt{x})$

$$-1 \le x - 2\sqrt{x} \le 1$$
$$x \ge 0$$

① $x - 2\sqrt{x} \ge -1$

$x - 2\sqrt{x} + 1 \ge 0 \Rightarrow t^2 - 2t + 1 \ge 0$

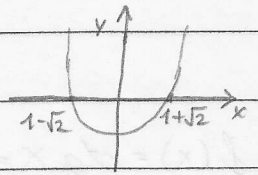
$t = \sqrt{x}$ $\qquad (t-1)^2 \ge 0 \quad \forall t \in \mathbb{R}$

$\boxed{x \ge 0}$

② $x - 2\sqrt{x} \le 1$

$x - 2\sqrt{x} - 1 \le 0 \Rightarrow t^2 - 2t - 1 \le 0$

$t = \sqrt{x}$ $\qquad t^2 - 2t - 1 = 0 \Rightarrow t_{1,2} = 1 \pm \sqrt{2}$



$t \in [1-\sqrt{2}, 1+\sqrt{2}]$

$t = \sqrt{x} \rightarrow \sqrt{x}$ je pozitivan pa: $t \in [0, 1+\sqrt{2}]$

$\sqrt{x} = 1+\sqrt{2} \quad /^2$

$x = 3 + 2\sqrt{2} \Rightarrow x \in [0, 3+2\sqrt{2}]$

„1." ∩ „2." ⇝ $x \in [0, 3+2\sqrt{2}]$

Dz - jesenski 2013 (arccos ovo gradiva)

- dekanski 2013 (injektivnost)

HIPERBOLIČKE FUNKCIJE

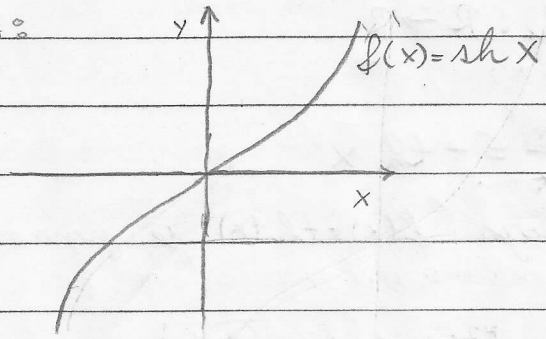
⇒ SINUS HIPERBOLIČKI:

$$f(x) = \operatorname{sh} x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{sh}: R \to R$$

- $\operatorname{sh}(-x) = \frac{e^{-x} - e^{-(-x)}}{2} = -\frac{e^x - e^{-x}}{2} = -\operatorname{sh}(x) \quad \forall x \in D(f)$

↳ funkcija $f(x) = \operatorname{sh} x$ je neparna, pa je dovoljno promatrati na $[0, \infty\rangle$

- $\operatorname{sh}(0) = 0$
- $x \to \infty$, $\operatorname{sh} x = \frac{e^x - e^{-x}}{2} = \frac{e^x - \overbrace{\frac{1}{e^x}}^{\to 0}}{2} \longrightarrow +\infty$
- SKICA:



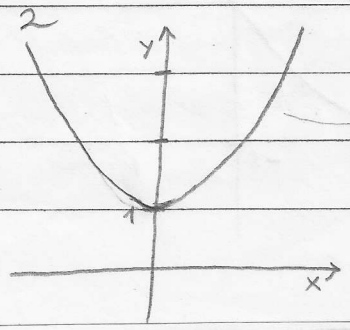
⇒ KOSINUS HIPERBOLIČKI:

$$f(x) = \operatorname{ch} x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{ch}: R \to R$$

- $\operatorname{ch}(-x) = \frac{e^{-x} + e^x}{2} = \operatorname{ch} x \quad \forall x \in D(f)$

↳ funkcija $f(x) = \operatorname{ch} x$ je parna, pa je dovoljno promatrati na $[0, \infty\rangle$

- $\operatorname{ch}(0) = 1 \Rightarrow \operatorname{Im} f = [1, \infty\rangle$
- $x \to \infty$, $\operatorname{sh} x = \frac{e^x + e^{-x}}{2} \longrightarrow +\infty$
- SKICA:



→ mjo parabola, ove se: „LANČANICA"

⇒ obja$\check{s}$njenje imena „hiperbolične" funkcije:

• $ch^2x - sh^2x = 1$

$ch^2t - sh^2t = 1$

$x = ch\,t \qquad y = sh\,t$

$$x^2 - y^2 = \left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 = \ldots = 1$$

⇒ ADICIJSKE I OSTALE FORMULE: knjižica: str. 15

↳ $ch(2x) = ch^2x + sh^2x$

tu je minus kod trigonometrijskih

⇒ TANGENS HIPERBOLIČKI:

$$f(x) = th(x) = \frac{sh\,x}{ch\,x}, \qquad th : R \to R$$

• $$th(-x) = \frac{sh(-x)}{ch(-x)} = \frac{-sh\,x}{ch\,x} = -th\,x$$

↳ funkcija $f(x) = th(x)$ je neparna

• $th(0) = 0$

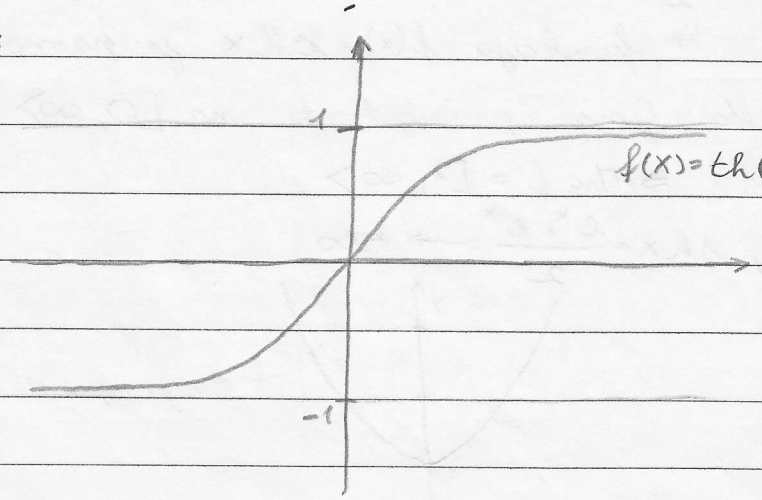
• $$th\,x = \frac{e^{2x} - 1}{e^{2x} + 1} \overset{t = e^x}{=} \frac{t^2 - 1}{t^2 + 1} = \frac{t^2 + 1 - 2}{t^2 + 1} = 1 - \frac{2}{t^2 + 1}$$

$x \to \infty$

$th\,x \to \infty$

• $|th\,x| < 1, \quad \forall x \in R$

• SKICA:



ZAD: Dokaž da je $f(x) = \operatorname{th} x$ rastuća na $[0, \infty\rangle$!

↳ RJ: $x, y > 0$ i gledamo $\operatorname{th}(x+y) - \operatorname{th} y = \ldots = \frac{\operatorname{sh} x}{\operatorname{ch}(x+y) \cdot \operatorname{ch} y} > 0$

$\forall x \in [0, \infty\rangle$

⇒ KOTANGENS HIPERBOLIČKI:

$$\operatorname{cth} x = \frac{\operatorname{ch} x}{\operatorname{sh} x}, \quad \operatorname{cth}: \mathbb{R} \setminus \{0\} \to \mathbb{R}$$

- $\operatorname{cth}(-x) = -\operatorname{cth} x, \quad \forall x \in D(f)$
- $\operatorname{cth} x = \frac{1}{\operatorname{th} x}; \quad |\operatorname{cth} x| > 1$

  $\operatorname{Im}(\operatorname{cth} x) = \mathbb{R} \setminus [-1, 1]$

- $x \to 0^+$

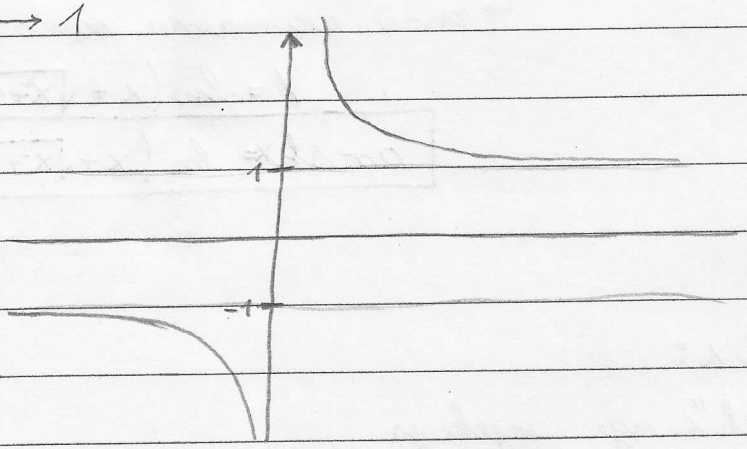
  ponaša se kao $\frac{1}{0^+}$

  $$\operatorname{cth} x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{e^{2x} + 1}{e^{2x} - 1} \to +\infty$$

- $x \to +\infty$

  $\operatorname{cth} x \to 1$

- SKICA:



Dz - 1MI 2012 (nešto s $\operatorname{ch} x$)

AREA FUNKCIJE

⇒ inverzne funkcije hiperboličkih funkcija

⇒ AREA sh x :

$sh : R \to R$ je bijekcija, pa $\exists\, sh^{-1} : R \to R$ (naziv: arsh)

$$y = \frac{e^x - e^{-x}}{2} \; ; \quad x = \frac{e^y - e^{-y}}{2}$$

! izrazi „y"

$$e^{2y} - 2x e^y - 1 = 0 \; ; \quad e^y = t$$

$$t^2 - 2x t - 1 = 0$$

$$t_{1,2} = \frac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$$

$$e^y = x \pm \sqrt{x^2+1} \Rightarrow$$

↳ kako je $e^y > 0$, $\forall y \in R$ onda mora vrijediti $x \pm \sqrt{x^2+1} > 0$

↳ znači uzimamo samo „+":

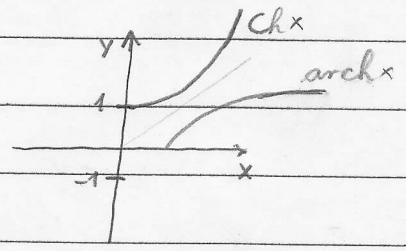
$$y = \ln\left(x + \sqrt{x^2+1}\right)$$

$$\boxed{ar\,sh\,x = \ln\left(x + \sqrt{x^2+1}\right)}$$

⇒ AREA ch x :

„ch" nije injekcija

↳ restrikcija: $f|_{[0,\infty\rangle} \equiv Ch$



$$Ch : [0, \infty\rangle \to [1, \infty\rangle$$

$$arch : [1, \infty\rangle \to [0, \infty\rangle$$

$$y = \frac{e^x + e^{-x}}{2} \; ; \quad x = \frac{e^y + e^{-y}}{2}$$

$$2x = e^y + \frac{1}{e^y}$$

$$e^{2y} - 2x e^y + 1 = 0$$

$$e^y = t$$

$$t^2 - 2xt + 1 = 0 \Rightarrow t_{1,2} = \frac{2x \pm \sqrt{4x^2-4}}{2} = x \pm \sqrt{x^2-1}$$

$x \geq 1$ → uvjet iz restrikcije

$$e^y = x \pm \sqrt{x^2-1} \;/\; \ln$$

↳ gledamo $x - \sqrt{x^2-1}$: vrijedi li : $0 < x - \sqrt{x^2-1} \leq 1$

$x - \sqrt{x^2-1} > 0$ $\qquad$ $x - \sqrt{x^2-1} \leq 1$

$x > \sqrt{x^2-1} \;/^2$ $\qquad$ $x - 1 \leq \sqrt{x^2-1} \;/^2 \; (x \geq 1)$

$0 > -1$ ✓ $\qquad$ $x^2 - 2x + 1 \leq x^2 - 1$

$1 \leq x$ ✓ $(x \geq 1)$

to ne želimo jer nam „ln" tada mijenja smisao / na nas zanima samo : $e^y = x + \sqrt{x^2-1}$

$$e^y = x + \sqrt{x^2-1} \;/\; \ln$$

$$y = \ln\left(x + \sqrt{x^2-1}\right)$$

$$\boxed{\operatorname{ar\,ch} x = \ln\left(x + \sqrt{x^2-1}\right)}$$

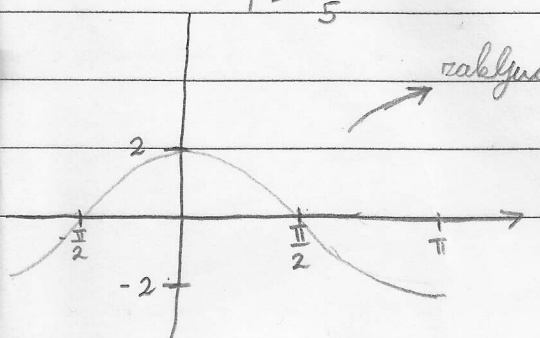
⇒ za vježbu izvedi : a) $\operatorname{ar\,th} x$

b) $\operatorname{ar\,cth} x$

↳ također i skiciraj funkcije

ZAD : Odredi najveći $a \in \mathbb{R}$ tako da $f : [0, a] \to \mathbb{R}$, $f(x) = 2 + 2\cos(5x)$ bude injekcija. Za takav „a" odredi $\mathrm{Im}\, f$!

$$T = \frac{2\pi}{5}$$

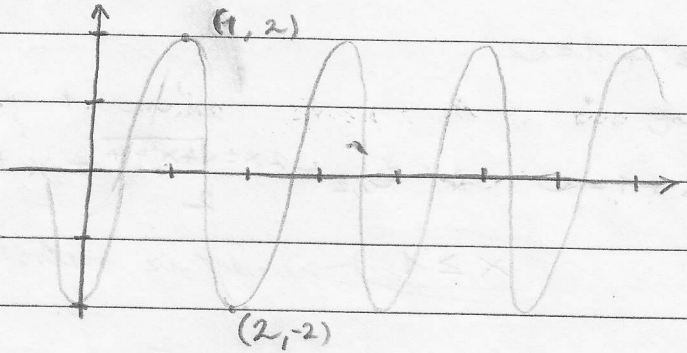


zaključak : funkcija je bijekcija na polovini perioda

$$a = \frac{T}{2} = \frac{\pi}{5}$$

$$\mathrm{Im}\, f\, [0, 4]$$

ZAD: Odredi jednadžbu sinusoide sa slike:



$T = 2$ $\quad \omega = \frac{2\pi}{T}$ $\quad \omega = \pi$

$A = 2$

$Y = A \sin(\omega x + \varphi)$

↳ „φ" ćemo izračunat preko uvrštavanja točaka: $(1, 2)$

$$2 = 2 \sin(\pi \cdot 1 + \varphi)$$

$$\vdots$$

$$\varphi = -\frac{\pi}{2} + 2k\pi \; ; \; k \in \mathbb{Z}$$

$$\varphi = -\frac{\pi}{2} \quad \Rightarrow \quad Y = 2 \sin\left(\pi \cdot x - \frac{\pi}{2}\right)$$

NIZOVI

PR: Imamo 5g materijala poluraspada nakon 1 tjedna. Koliko materijala ima nakon „n" tjedana?

$a_1 = 5$

$a_2 = \frac{a_1}{2} = \frac{5}{2}$

$a_3 = \frac{a_1}{2^2} = \frac{5}{4}$

$\vdots$

$a_n = \frac{a_1}{2^{n-1}}$; $n \in \mathbb{N}$ $\Rightarrow$ Pitanje: $\lim_{n\to\infty} a_n = ?$

$a_n \to 0$, $n \to \infty$

DEF: Niz realnih brojeva je funkcija $a: \mathbb{N} \to \mathbb{R}$

$a(n) = a_n$

$\Rightarrow$ je li niz u polaznom primjeru:

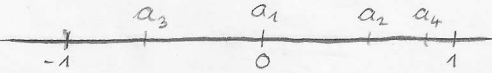
a) MONOTON - da, padajući je

b) OMEĐEN, da $(0 < a_n \leq 5, \forall n \in \mathbb{N})$

DEF: Niz je omeđen kada ima i gornju i donju među, tj. ako $\exists m, M \in \mathbb{R}$ takvi da $\forall n \in \mathbb{N}$ vrijedi $m \leq a_n \leq M$

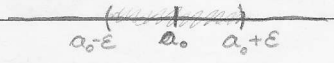
PR: $a_n = (-1)^n \cdot \frac{n-1}{n}$ $\to$ niz: $0, \frac{1}{2}, (-1)\cdot\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \frac{5}{6}, -\frac{6}{7}, \ldots$

↳ skica:



↳ omeđen, ali ne monoton

$\Rightarrow$ OKOLINA REALNOG BROJA (ε):



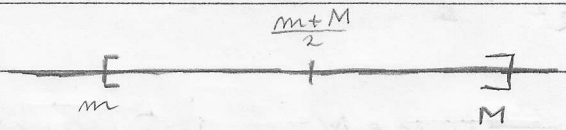
↳ to je $\{a \in \mathbb{R} : |a - a_0| < \varepsilon\}$, $\varepsilon > 0$

⇒ GOMILIŠTE NIZA:

↳ realan broj A je gomilište niza (a_n) ako se unutar svake okoline broja A nalazi beskonačno mnogo članova niza (a_n).

⇒ Teorem: svaki omeđeni niz realnih brojeva ima barem jedno gomilište

↳ skica dokaza:



• barem jedan od podintervala $[m, \frac{m+M}{2}]$, $[\frac{m+M}{2}, M]$ sadrži beskonačno mnogo članova niza

↳ dalje polovimo te podintervale, te presjek svih tih podintervala je gomilište

KONVERGENCIJA NIZA REALNIH BROJEVA

⇒ intuitivna skrtežnja konvergencije:

↳ niz (a_n) konvergira prema $L \in \mathbb{R}$, onda se izvan svake okoline od L nalazi samo konačno mnogo članova niza

⇒ LIMES NIZA (a_n):

↳ oznaka: $\lim_{n\to\infty} a_n = L$

↳ L je limes niza kada „n" teži u beskonačno ako $\forall \varepsilon > 0$ $\exists n_0 \in \mathbb{N}$ takav da $\forall n \geq n_0$ vrijedi nejednakost $|a_n - L| < \varepsilon$



PR: $\lim_{n\to\infty} \frac{1}{n} = 0$

↳ uzmemo $\varepsilon > 0$

↳ želimo naći $n_0 \in \mathbb{N}$ takav da $\forall n \geq n_0$ vrijedi

$$\left|\frac{1}{n} - 0\right| < \varepsilon \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow n > \frac{1}{\varepsilon}$$

$$n_0 > \frac{1}{\varepsilon}$$

⇒ osnovni zaključci o limesu niza:

- konvergentan niz je omeđen
- konvergentan niz ima samo jedan limes
- $(a_n), (b_n), (c_n)$ te postoji $n_0 \in \mathbb{N}$ takav da vrijedi $a_n \leq c_n \leq b_n$, $\forall n \geq n_0$ te „$\lim_{n\to\infty} a_n = L$" i „$\lim_{n\to\infty} b_n = L$", onda je i „$\lim_{n\to\infty} c_n = L$"

⇒ operacije s nizovima:

$$(a_n) \Rightarrow \lim_{n\to\infty} a_n = \alpha \quad ; \quad (b_n) \Rightarrow \lim_{n\to\infty} b_n = \beta$$

$$\lim_{n\to\infty} (a_n \overset{\pm}{\underset{\cdot}{\cdot}} b_n) = \lim_{n\to\infty} a_n \overset{\pm}{\underset{\cdot}{\cdot}} \lim_{n\to\infty} b_n$$

⇒ divergentni nizovi su oni koji imaju limes u beskonačno

$$\exists \varepsilon > 0, \forall n_0 \in \mathbb{N}, \exists n \geq n_0, |a_n - L| \geq \varepsilon$$

ZAD: $\lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) - \lim_{n\to\infty} (\sqrt{n+1} - \sqrt{n}) \cdot \frac{\sqrt{n+1}+\sqrt{n}}{\sqrt{n+1}+\sqrt{n}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$

ZAD: $\lim_{n\to\infty} \underbrace{\frac{3^n}{n!}}_{a_n} = ?$

$$a_{n+1} = \frac{3^{n+1}}{(n+1)!} = \frac{3}{n+1} \cdot \frac{3^n}{n!} = \frac{3}{n+1} \cdot a_n \quad < 1, \forall n > 2$$

$\square$

$a_{n+1} < a_n$ ⇒ znači da je (a_{n+1}) konvergentan

$$\lim_{n\to\infty} \left(\frac{3}{n+1} \cdot a_n\right) = \lim_{n\to\infty} \left(\frac{3}{n+1}\right) \cdot \lim_{n\to\infty} a_n = 0 \cdot L = 0$$

ZAD: $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+k}}$ ⇒ Forest „endvič-teorem"

$$a_n \leq S_n \leq b_n$$

$$\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \ldots + \frac{1}{\sqrt{n^2+n}} \longrightarrow S_n$$

ograničimo ga odozgo

$$\frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{\sqrt{n^2+n}} \leq \overbrace{\frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{\sqrt{n^2+1}}}^{\frac{n}{\sqrt{n^2+1}}}$$

ograničimo ga odozdo

$$\frac{1}{\sqrt{n^2+1}} + \ldots + \frac{1}{n^2+n} \geq \underbrace{\frac{1}{\sqrt{n^2+n}} + \frac{1}{\sqrt{n^2+n}} + \ldots + \frac{1}{\sqrt{n^2+n}}}_{\frac{n}{\sqrt{n^2+n}}}$$

↳ stoga: $\frac{n}{\sqrt{n^2+n}} \leq S_n \leq \frac{n}{\sqrt{n^2+1}}$

$\lim_{n\to\infty} \frac{n}{\sqrt{n^2+n}} = 1$; $\lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = 1$

↳ teorem o sendviču: $\lim_{n\to\infty} S_n = 1$

ZAD: $\lim_{n\to\infty} \left(\sqrt[3]{n^2-n^3}+n\right) = \lim_{n\to\infty} \left(n-\sqrt[3]{n^3-n^2}\right) \cdot \frac{n^2+n\sqrt[3]{n^3-n^2}+\sqrt[3]{(n^3-n^2)^2}}{n^2+n\sqrt[3]{n^3-n^2}+\sqrt[3]{(n^3-n^2)^2}}$

$= \lim_{n\to\infty} \frac{n^3-(n^3-n^2)}{n^2+n\cdot\sqrt[3]{n^3-n^2}+\sqrt[3]{(n^3-n^2)^2}} = \lim_{n\to\infty} \frac{n^2}{n^2+n\cdot\sqrt[3]{n^3-n^2}+\sqrt[3]{(n^3-n^2)^2}} \quad / : n^2$

$= \frac{1}{3}$

ZAD: $\lim_{n\to\infty} \frac{\sqrt[3]{n^2}\cdot \sin(n!)}{n+1}$ → nema limes

$-1 \leq \sin(n!) \leq 1$

$\lim_{n\to\infty} \frac{\sqrt[3]{n^2}}{n+1} = \lim_{n\to\infty} \frac{\sqrt[3]{\frac{1}{n}}}{1+\frac{1}{n}} = 0$ → 0

↳ rješenje nije nula već je rješenje:
produkt niza koj konvergira
prema nuli i ograničenog niza

ZAD: $\lim_{n\to\infty} \frac{2n\cdot\sin\left(\frac{\pi}{2}\right)}{n+1} = 2\lim_{n\to\infty}\left(\frac{n}{n+1}\cdot\sin\left(\frac{n\pi}{2}\right)\right)$ → ne postoj !!!

→ teži u 1

ZAD: $\lim_{n\to\infty} \cos^n(n\pi) \Rightarrow$ limes ne postoj

ZAD: $\lim_{n\to\infty} \left(1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}+\ldots+\frac{(-1)^{n-1}}{3^{n-1}}\right) = \lim_{n\to\infty}\left(1\cdot\frac{1-\left(-\frac{1}{3}\right)^n}{1-\left(-\frac{1}{3}\right)}\right) = \frac{3}{4}$

$a_1 = 1$
$a_2 = -\frac{1}{3}$ → $q = -\frac{1}{3}$
$a_3 = \frac{1}{9}$
$\vdots$

$a_n = \frac{(-1)^{n-1}}{3^{n-1}} \quad n \geq 1$

$S_n = a_1 \cdot \frac{1-q^n}{1-q}$

$S_n = a_1+a_2+\ldots+a_n$
$= a_1+a_1q+a_1q^2+\ldots+a_1q^{n-1}$

⇒ VAŽAN LIMES: $\boxed{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e}$

ZAD: $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}} = \lim_{n\to\infty}\left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{\frac{1}{2}} = \sqrt{e}$

ZAD: $\lim_{n\to\infty}\left(\frac{n}{n-1}\right)^n = \lim_{n\to\infty}\left(\frac{n-1+1}{n-1}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n-1}\right)^n = \lim_{n\to\infty}\left(\left(1+\frac{1}{n-1}\right)^{n-1}\right)^{\frac{n}{n-1}}$

$= e^{\lim_{n\to\infty}\left(\frac{n}{n-1}\right)} = e$

ZAD: $\lim_{n\to\infty}\left(\frac{3n}{n+1}\right)^{\frac{n}{2n+1}} = \lim_{n\to\infty}\left(\frac{3n}{n+1}\right)^{\lim_{n\to\infty}\left(\frac{n}{2n+1}\right)} = 3^{\frac{1}{2}} = \sqrt{3}$

ZAD: $\lim_{n\to\infty} n\left[\ln(n+3)-\ln n\right] = \lim_{n\to\infty} n\left(\ln\frac{n+3}{n}\right) = \lim_{n\to\infty} n\cdot\ln\left(1+\frac{3}{n}\right)$

$= \lim_{n\to\infty}\ln\left(1+\frac{3}{n}\right)^n = \lim_{n\to\infty}\left[\ln\left(1+\frac{1}{\frac{n}{3}}\right)^{\frac{n}{3}}\right]^3 = \ln e^3 = 3$

limes može „ući" u neprekidnu funkciju

ZAD: $\lim_{n\to\infty}\left(1-\frac{1}{3n}\right)^n = \lim_{n\to\infty}\left(\frac{3n-1}{3n}\right)^n = \lim_{n\to\infty}\left(\frac{1}{\frac{3n}{3n-1}}\right)^n = \lim_{n\to\infty}\left(\frac{1}{\frac{3n-1+1}{3n-1}}\right)^n$

$= \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{3n-1}\right)^n} = \lim_{n\to\infty}\left(1+\frac{1}{3n-1}\right)^{-n} = \lim_{n\to\infty}\left[\left(1+\frac{1}{3n-1}\right)^{3n-1}\right]^{\frac{-n}{3n-1}}$

$= e^{\lim_{n\to\infty}\left(\frac{-n}{3n-1}\right)} = e^{-\frac{1}{3}}$

⇒ zaključak: korisna formula: $\boxed{\lim_{n\to\infty}\left(1+\frac{k}{n}\right)^n = e^k}$ $k\in R$

DZ: a) $\lim_{n\to\infty}\left(\frac{n+\frac{1}{3}}{n-\frac{2}{3}}\right)^{\frac{n}{3}}$ $R_J = e^{\frac{1}{3}}$

b) $\lim_{n\to\infty}\left(\frac{n^2+\frac{1}{3}}{n^2-\frac{2}{3}}\right)^{\frac{n}{3}}$ $R_J = 1$

EKVIVALENTNE NEIZMJERNO VELIKE VELIČINE

(a_n) ; (b_n)

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \infty$; $\lim_{n\to\infty} \frac{a_n}{b_n} = C \qquad C \in R$

↳ (a_n) ; (b_n) su neizmjerno velike veličine istog reda

↳ ako je $C = 1$, onda su te veličine ekvivalentne

PR: $\lim_{n\to\infty} \frac{n + \sqrt[3]{n^4+1}}{n\sqrt{n}+1} \;/: n^{\frac{3}{2}} \;/: n^{\frac{3}{2}} = \lim_{n\to\infty} \frac{\frac{1}{n^{\frac{1}{2}}} + \sqrt[6]{\frac{(n^4+1)^2}{n^9}}}{1 + \frac{1}{n\sqrt{n}}} = 0$

↳ 2. način:

$$\lim_{n\to\infty} \frac{n + \sqrt[3]{n^4+1}}{\sqrt[3]{n}+1} = \lim_{n\to\infty} \frac{\frac{n + \sqrt[3]{n^4+1}}{n^{\frac{4}{3}}} \cdot n^{\frac{4}{3}}}{\frac{n\sqrt{n}+1}{n\sqrt{n}} \cdot n^{\frac{3}{2}}} = \lim_{n\to\infty} \frac{n^{\frac{4}{3}}}{n^{\frac{3}{2}}} = \lim$$

($\frac{n + \sqrt[3]{n^4+1}}{n^{\frac{4}{3}}} \to 1$, $\frac{n\sqrt{n}+1}{n\sqrt{n}} \to 1$)

$$= \lim_{n\to\infty} n^{-\frac{1}{6}} = 0$$

DZ: $\lim_{n\to\infty} \frac{n^2+1}{n^a + \sqrt{n^4+1}}$ $\qquad R_f = \begin{cases} \frac{1}{2}, & a=2 \\ 0, & a>2 \\ 1, & a<2 \end{cases}$

MONOTONI NIZOVI

- rastući: $n < m \Rightarrow a_n \leq a_m$
- padajući: $n \leq m \Rightarrow a_n \geq a_m$

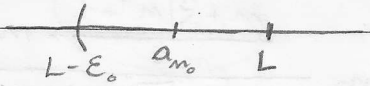
↳ možemo li raključiti je li niz konvergentan bez računanja limesa?

⇒ TEOREM: ako je niz realnih brojeva monoton i omeđen, onda je on konvergentan

↳ dokaz ⇒ (a_n) je rastući, omeđen

⇒ S je skup svih vrijednosti niza (a_n), te je taj skup omeđen odozgo, pa postoji najmanja gornja međa L što znači:

$$\forall \varepsilon > 0;\ \exists n_0 \in \mathbb{N}\quad a_{n_0} > L - \varepsilon_0$$



⇒ nadalje, kako je (a_n) rastući niz, onda:

$$\forall n \geq n_0 \text{ vrijedi } a_n > a_{n_0} > L - \varepsilon$$

monoton + omeđen = konvergentan

PITANJA: Da li je svaki omeđen niz konvergentan?

↳ NE, npr. „sin n"

Može li niz biti konvergentan, a da nije monoton?

↳ DA, npr. $\frac{(-1)^n}{n} \to 0$ //

ZAD: $(a_n) \Rightarrow a_1 = 1 \quad a_{n+1} = \frac{a_n(a_n+1)}{3}, \; n \geq 1$

↳ dokaži da je niz konvergentan, te odredi limes ∇!

$a_1 = 1, \; a_2 = \frac{2}{3}, \; a_3 = \frac{\frac{2}{3}(\frac{2}{3}+1)}{3} = \frac{10}{27}, \ldots\ldots$

↳ niz pada → dokaz indukcijom ∇

① ograničenost odozdo treba pretpostavit: $a_n \geq 0, \; \forall n \in \mathbb{N}$

- baza indukcije: $n=1, \; a_1 = 1 \geq 0$
- pretpostavka da za neki $n \in \mathbb{N}$ vrijedi $a_n \geq 0$
- dokažimo da je onda i $a_{n+1} \geq 0$

$a_n \geq 0$ — TRIVIJALNA INDUKCIJA

$a_n^2 + a_n \geq 0$

$\frac{a_n^2 + a_n}{3} = a_{n+1} \geq 0$

↳ pokazali smo da je $a_n \geq 0 \; \forall n \in \mathbb{N}$ //

② monotono padajući: $a_n \geq a_{n+1}, \; \forall n \in \mathbb{N}$

- baza indukcije: $n=1, \; a_1 = 1 \geq \frac{2}{3} = a_2$
- pretpostavka da za neki $n \in \mathbb{N}$ vrijedi: $a_n \geq a_{n+1}$
- dokažimo da je $a_{n+1} \geq a_{n+2}$

↳ krenimo od pretpostavke:

$a_n \geq a_{n+1} \quad / ^2$ (jer je $a_n \geq 0 \; \forall n$)

$a_n^2 \geq a_{n+1}^2$

$a_n^2 + a_n \geq a_{n+1}^2 + a_n \geq a_{n+1}^2 + a_{n+1}$

koristimo postavku

$\frac{a_n(a_n+1)}{3} \geq \frac{a_{n+1}(a_{n+1}+1)}{3}$

$a_{n+1} \geq a_{n+2}$ //

③ pokazali smo da je niz omeđen i monoton, pa znamo da je konvergentan i ima limes: $L = \lim_{n\to\infty} a_n = \lim_{n\to\infty} a_{n+1}$

$\lim_{n\to\infty}(a_{n+1}) = \frac{\lim_{n\to\infty} a_n^2 + \lim_{n\to\infty} a_n}{3}$

$L = \frac{L^2 + L}{3} \Rightarrow L^2 - 2L = 0 \rightarrow L_1 = 0 \quad L_2 = 2$ → nema smisla jer $a_1 = 1$ i niz je padajući

$L(L-2) = 0$

$\boxed{L = 0}$

⇒ dobar je trik prvo napraviti 3. do zadatka kako bi lakše pretpostavil među, no ne zaboravi dokazat omeđenost i monotonost niza

↳ NAPOMENA: možemo pogriješit ! - važno je prije računanja limesa dokazat konvergentnost

npr. niz: $a_1 = 1$; $a_{n+1} = 1 - a_n$

1, 0, 1, 0, 1, 0, ...

↳ nije konvergentan, a za među bi dobil $L = \frac{1}{2}$

ZAD: $a_1 = 1$ Dokaži konvergenciju i odredi limes!

$$a_{n+1} = 1 - \frac{1}{\sqrt{a_n + 1}}$$

$$a_1 = 1 \; ; \; a_2 = 1 - \frac{1}{\sqrt{2}} \; ; \; a_3 = 1 - \frac{1}{\sqrt{1 - \frac{1}{\sqrt{2}} + 1}} = 1 - \frac{1}{\sqrt{2 - \frac{1}{\sqrt{2}}}}$$

① (a_n) je padajući (mat. indukcija)

② omeđen odozdo (mat indukcija)

③ $L = \lim_{n \to \infty} a_n$

② Tvrdnja: niz (a_n) je ograničen odozdo sa 0

- baza: $n=1 \quad a_1 = 1 > 0$
- pretpostavka: za neki $n \in \mathbb{N}$ vrijedi $a_n > 0$
- pokažimo: $a_{n+1} > 0$

$$a_{n+1} = 1 - \frac{1}{\sqrt{a_n + 1}} \overset{?}{>} 0 \qquad a_n > 0$$

$$a_n + 1 > 1 \quad / \sqrt{\ }$$

$$\sqrt{a_n + 1} > 1$$

$$\frac{1}{\sqrt{a_n+1}} < 1 \Rightarrow \underbrace{1 - \frac{1}{\sqrt{a_n+1}}}_{a_{n+1}} > 0$$

↳ prema principu matematičke indukcije, dobacali smo tvrdnju

③ (a_n) je omeđen i monoton, što znači da je konvergentan, pa $\exists L = \lim_{n\to\infty} a_n$

$$a_{n+1} = 1 - \frac{1}{\sqrt{a_n + 1}} \quad / \lim_{n\to\infty}$$

$$L = 1 - \frac{1}{\sqrt{L+1}} \quad (0 \leq L < 1) \quad \square$$

$$\frac{1}{L+1} = 1 - 2L + L^2$$

$$L(L^2 - L - 1) = 0$$

$$L_1 = 0 \; ; \quad L_2 = \frac{1-\sqrt{5}}{2} \; ; \quad L_3 = \frac{1+\sqrt{5}}{2}$$

↳ „L_1" je rješenje ∇

ZAD: Dokažite da niz $a_n = \sum_{k=0}^{n} \frac{1}{k!}$ konvergira?

$$a_n = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \ldots + \frac{1}{n!}$$

- niz je monotono rastuć? $a_{n+1} > a_n \quad \forall n \in \mathbb{N}$

$$a_{n+1} - a_n = \sum_{k=0}^{n+1} \frac{1}{k!} - \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{(n+1)!} > 0$$

↳ DA! Niz je monotono rastuć

- niz je omeđen odozgo?

$$a_n = 1 + 1 + \frac{1}{1\cdot 2} + \frac{1}{1\cdot 2\cdot 3} + \frac{1}{1\cdot 2\cdot 3\cdot 4} + \ldots + \frac{1}{1\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot n}$$

$$a_n \leq 1 + 1 + \frac{1}{2} + \frac{1}{2\cdot 2} + \frac{1}{2\cdot 2\cdot 2} + \ldots + \frac{1}{2\cdot 2\cdot 2\cdot \ldots \cdot n}$$

$$a_n \leq 1 + \underbrace{1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots + \frac{1}{2^{n-1}}}$$

$$1 + q + q^2 + \ldots + q^{n-1} \quad ; \quad q = \frac{1}{2}$$

$$a_n \leq 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{1 - \frac{1}{2}} = 1 + \frac{1 - \left(\frac{1}{2}\right)^n}{\frac{1}{2}}$$

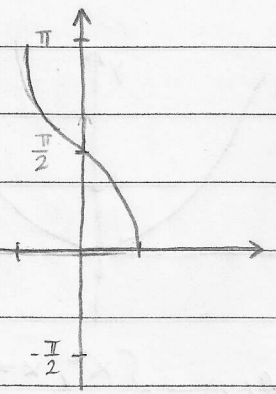
$$a_n \leq 1 + 2\left(1 - \left(\frac{1}{2}\right)^n\right)$$

$$a_n \leq 3 - \frac{1}{2^{n-1}} < 3 \quad \forall n \in \mathbb{N}$$

ZADATAK „malo funkcija"

↳ odredi sliku i domenu funkcije $f(x)=\arccos\left(\frac{1}{e^x+2}\right)$

↳ odredi i inverznu funkciju



$$-1 \le \frac{1}{e^x+2} \le 1$$

$$e^x+2 \neq 0 \ \forall x \in \mathbb{R} \Rightarrow D_f = \mathbb{R}$$

$$e^x+2 \in \langle 2, +\infty \rangle$$

$$0 < \frac{1}{e^x+2} < \frac{1}{2} \ / \arccos \quad (\text{PADAJUĆA?})$$

$$\arccos 0 > \arccos\left(\frac{1}{e^x+2}\right) > \arccos\frac{1}{2}$$

$$\arccos\left(\frac{1}{e^x+2}\right) \in \left\langle \frac{\pi}{3}, \frac{\pi}{2} \right\rangle$$

$$y = \arccos\frac{1}{e^x+2}$$

$$x = \arccos\left(\frac{1}{e^y+2}\right); \ x \in [0, \pi]$$

$$\cos x = \frac{1}{e^y+2}$$

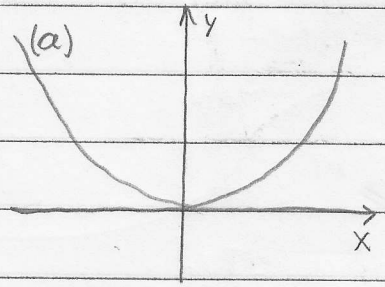
$$e^y+2 = \frac{1}{\cos x}$$

$$e^y = \frac{1}{\cos x} - 2 \quad (**) \Rightarrow \frac{1}{\cos x} - 2 > 0$$

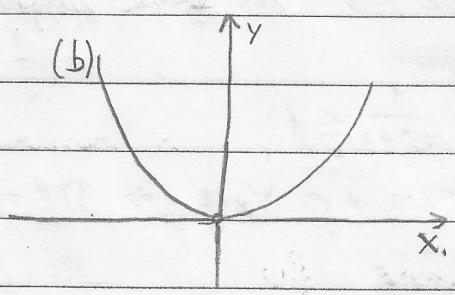
riješi

$$y = \ln\left(\frac{1}{\cos x} - 2\right)$$

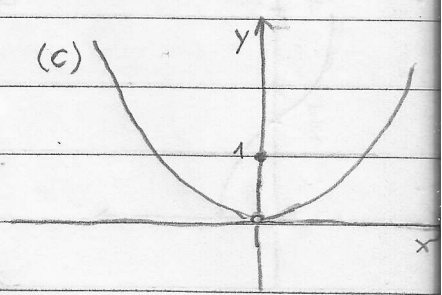
LIMES FUNKCIJE



$f(x) = x^2$



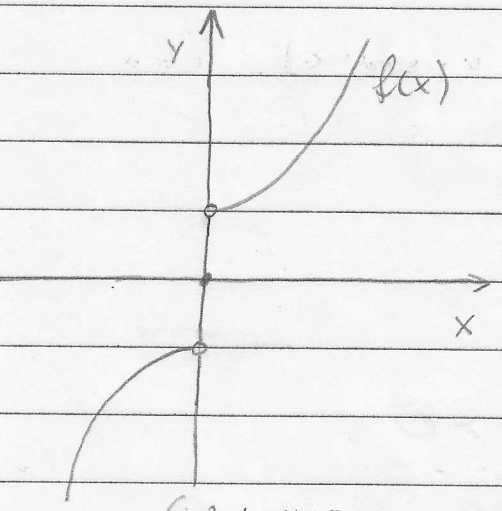
$f(x) = x^2 ; \; x \neq 0$



$$f(x) = \begin{cases} x^2, & x = 0 \\ 1, & x = 0 \end{cases}$$

$\lim_{x \to 0} f(x) = ?$

⇒ Intuitivna raklyučujemo: $\lim_{x \to a} f(x) = L$, ako su vrijednosti $f(x)$ „bliži" L (pri čemu je x iz neke okolne broja a



⇒ za ovu funkciju „$\lim_{x \to 0} f(x)$" ne postoji

↳ no, postoji limes zdesna:

$$\lim_{x \to 0^+} f(x) = 1$$

$$f(x) = \begin{cases} -x^2 - 1, & x < 0 \\ 0, & x = 0 \\ x^2 + 1, & x > 0 \end{cases}$$

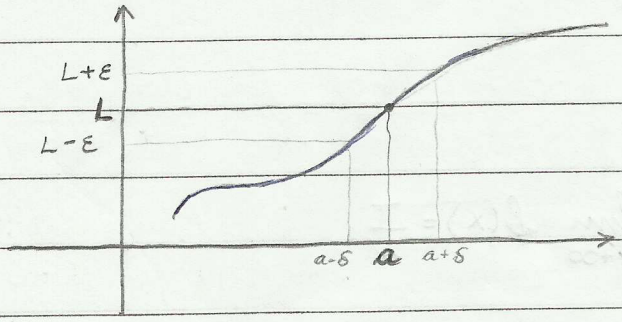
↳ postoji i limes slijeva:

$$\lim_{x \to 0^-} f(x) = 0$$

↳ funkcija je neprekidna ako (ove funkcije iz primjera):

- je definirana za $x = 0$
- vrijedi $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^+} f(x) = f(0)$

⇒ FORMALNA DEFINICIJA LIMESA:



⇒ intuicija: „ $f(x)$ može biti proizvoljno blizu L kada je x dovoljno blizu a"

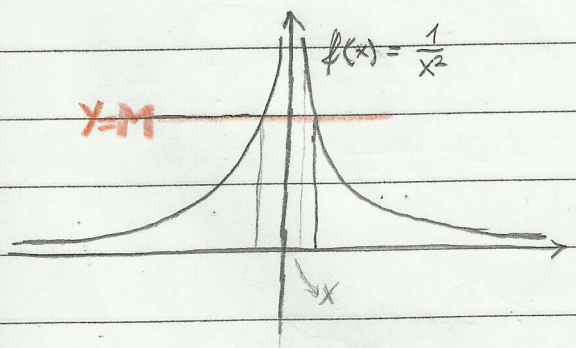
- udaljenost $f(x)$ do L: $|f(x)-L|$
  - ↳ za proizvoljni $\varepsilon>0$, možemo postići $|f(x)-L|<\varepsilon$
- za x kažemo da je dovoljno blizu a, ako je $|x-a|<\delta$

DEF: Za realan broj „L" kažemo da je limes funkcije „f" u točki „a", ako:

$\forall \varepsilon>0 \quad \exists \delta>0$ takav da $\forall x \in D(f)$, $x \neq a$ iz $|x-a|<\delta$ slijedi $|f(x)-L|<\varepsilon$

⇒ VRSTE LIMESA:

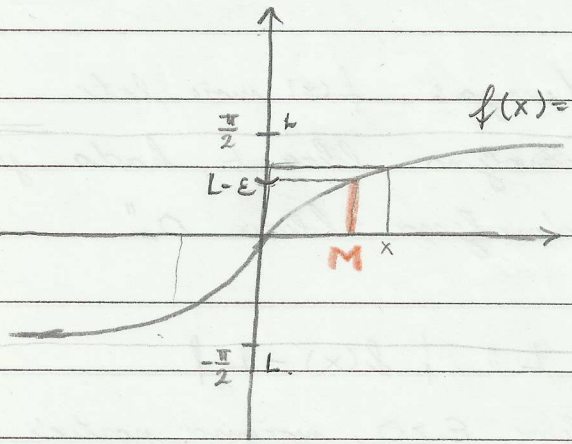
a) <u>beskonačni limes u konačnosti:</u>



$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)=+\infty$$

$\forall M>0 \quad \exists \delta>0$ takav da $\forall x \in D(f)$, $x \neq a$ iz $|x-a|<\delta$ slijedi $f(x)>M$

b) konačan limes u beskonačnosti:



$f(x) = \operatorname{arc\,tan} x$

$\lim_{x \to +\infty} f(x) = \frac{\pi}{2}$

$\lim_{x \to -\infty} f(x) = -\frac{\pi}{2}$

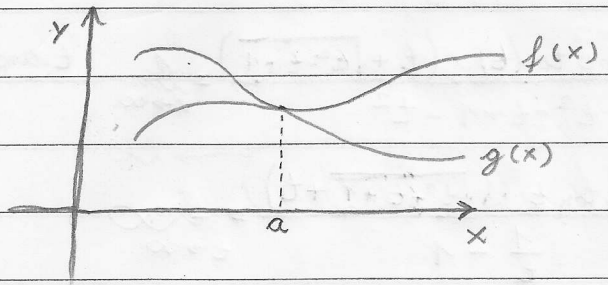
↳ $\lim_{x \to +\infty} f(x)$: $\forall \varepsilon > 0 \quad \exists M > 0$ takav da $\forall x \in D(f)$ iz $x > M$ slijed $|f(x) - L| < \varepsilon$

↳ $\lim_{x \to -\infty} f(x)$: $\forall \varepsilon > 0 \quad \exists M < 0$ takav da $\forall x \in D(f)$ iz $x < M$ slijed $|f(x) + L| < \varepsilon$

c) beskonačan limes u beskonačnost

RAČUNANJE LIMESA

⇒ vrlo korisan je „teorem o sendviču“:



$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = L$$

↳ ako vrijedi: $g(x) \le h(x) \le f(x) \quad \forall x$, onda vrijedi: $\lim_{x\to a} h(x) = L$ //

ZAD: $\lim_{x\to\infty} \frac{2x+1}{x+\sqrt{4x^2+3}} = \lim_{x\to\infty} \frac{2+\frac{1}{x}}{1+\sqrt{4+\frac{3}{x^2}}} = \frac{2}{3}$

ZAD: $\lim_{x\to-\infty} \frac{2x+1}{x+\sqrt{4x^2+3}}$ → supstitucija, $t=-x$;

$$\lim_{t\to\infty} \frac{-2t+1}{-t+\sqrt{4t^2+3}} \cdot \frac{\sqrt{4t^2+3}+t}{\sqrt{4t^2+3}+t} = \lim_{t\to\infty} \frac{(1-2t)(\sqrt{4t^2+3}+t)}{4t^2+3-t^2}$$

$$= \lim_{t\to\infty} \frac{(1-2t)(\sqrt{4t^2+3}+t)}{3t^2+3} \quad /:t^2 = \lim_{t\to\infty} \frac{(\frac{1}{t}-2)(\sqrt{4+\frac{3}{t}}+1)}{3+\frac{3}{t^2}} = -2$$

ZAD: a) $\lim_{x\to\infty} \frac{2x^3+1}{3x^2+3x+4} = \infty$

b) $\lim_{x\to-\infty} \frac{2x^3+1}{3x^2+3x+4}$ → supstitucija $t=-x$

$$\lim_{x\to\infty} \frac{-2t^3+1}{3t^2+3t+4} = -\infty$$

ZAD: a) $\lim_{x\to\infty} \frac{x \operatorname{arctg} x \,/:x}{x+\sqrt{x^2+x+1}\,/:x} = \lim_{x\to\infty} \frac{\operatorname{arctg} x}{1+\sqrt{1+\frac{1}{x}+\frac{1}{x^2}}} = \frac{\frac{\pi}{2}}{2} = \frac{\pi}{4}$

b) $\lim_{x\to-\infty} \frac{x \operatorname{arctg} x}{x+\sqrt{x^2+x+1}} = \left|\begin{array}{l} t=-x \\ x\to-\infty \\ t\to+\infty \end{array}\right| = \lim_{x\to\infty} \frac{-t \operatorname{arctg}(-t)}{-t+\sqrt{t^2-t+1}} = \lim_{x\to\infty} \frac{t \operatorname{arctg} t}{-t+\sqrt{t^2-t+1}}$

$= \lim_{x\to\infty} \frac{t \operatorname{arctg}(t)\cdot\left(t+\sqrt{t^2-t+1}\right)}{t^2-t+1-t^2} = \lim_{x\to\infty} \frac{t \operatorname{arctg}(t)\cdot(t+\sqrt{t^2-t+1})}{1-t}$

$= \lim_{t\to\infty} \frac{\operatorname{arctg} t \cdot \left(\sqrt{t^2-t+1}+t\right)}{\frac{1}{t}-1} = -\infty$,

($\sqrt{t^2-t+1}+t \to \infty$, $\frac{1}{t}-1 \to -1$)

ZAD: $\lim_{x\to\infty} \left(\frac{2x+\sin x}{x}\right)^{\frac{x+1}{x}} = 2^1 = 2$

$\frac{2x+\sin x}{x} = 2+\frac{1}{x}\cdot\sin x$ → ide u 2

→ ide u nulu.

VAŽNI LIMESI:

$$\lim_{n\to\infty} \sqrt[n]{n} = 1$$

$$\lim_{n\to\infty} \sqrt[n]{a} = 1 \;;\; a>0$$

OSNOVNI LIMESI:

① $\lim_{x\to 0} \frac{\sin x}{x} = 1$

↳ dokaz (pomoću „sendviča")

$f(x) = \frac{\sin x}{x} = f(-x)$ $\quad D_f = R \setminus \{0\}$

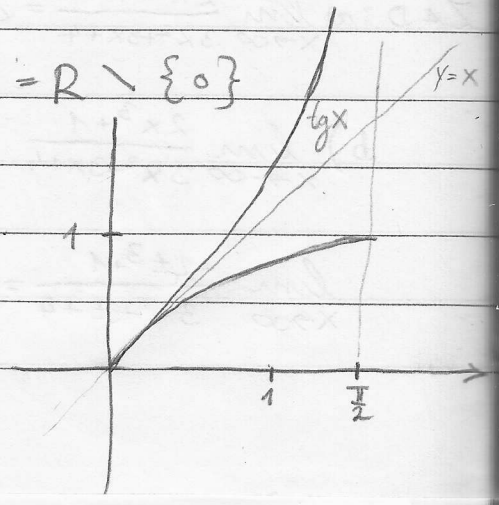
↳ Promatramo $x>0$, $x \in \langle 0, \frac{\pi}{2}]$

$\sin x < x < \operatorname{tan} x$

$\sin x < x < \frac{\sin x}{\cos x}$

$1 < \frac{x}{\sin x} < \frac{1}{\cos x}$

$\lim_{x\to 0} \frac{\sin x}{x} = 1$



želimo imat $\to \dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$

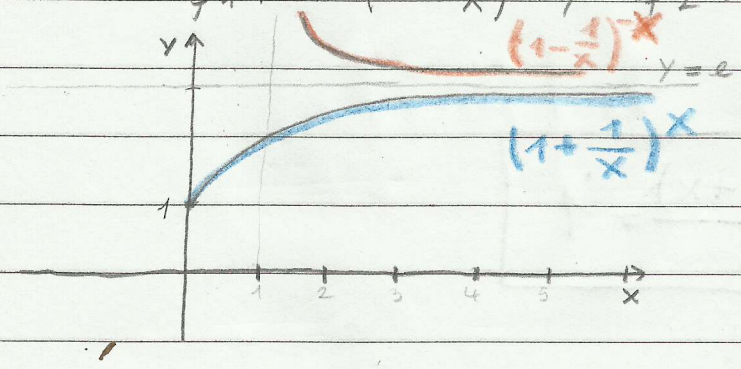
ZAD: a) $\lim\limits_{x\to\infty} \dfrac{x^2}{2x+1}\cdot\sin\left(\dfrac{1}{x}\right) = \lim\limits_{x\to\infty} \underbrace{\dfrac{x}{2x+1}}_{\to\frac{1}{2}} \cdot \overbrace{\dfrac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}}^{\to 1} = \dfrac{1}{2}$

ZAD: $\lim\limits_{x\to 0} \dfrac{\sin^3 x\cdot\cos(2x)}{x\cdot\sin^2(3x)} = \lim\limits_{x\to 0} \overbrace{\dfrac{\sin^3 x}{x^3}}^{\to 1}\cdot x^3\cdot\dfrac{1}{x}\cdot\overbrace{\dfrac{(3x)^2}{\sin^2(3x)}}^{\to 1}\cdot\dfrac{1}{(3x)^2}\cdot\overbrace{\cos(2x)}^{\to 1}$

$= \dfrac{1}{9}$

$$\boxed{② \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e, \quad \lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x = e}$$

• $f_1(x) = \left(1+\frac{1}{x}\right)^x$; $f_2(x) = \left(1-\frac{1}{x}\right)^{-x} = \left|\begin{matrix} t=-x \\ x\to\infty \\ t\to-\infty\end{matrix}\right| = \left(1+\frac{1}{t}\right)^t$



ZAD: $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+1}\right)^x \overset{[1^\infty]}{=} \lim\limits_{x\to\infty}\left(\dfrac{x+1-1}{x+1}\right)^x = \lim\limits_{x\to\infty}\left[\underbrace{\left(1+\dfrac{-1}{x+1}\right)^{x+1}}_{\to e^{-1}}\right]^{\overbrace{\frac{x}{x+1}}^{\to 1}} = e^{-1}$

↳ 2. način:

$$\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty}\frac{1}{\left(1+\frac{1}{x}\right)^x} = \frac{1}{e}$$

ZAD: $\lim\limits_{x\to\infty}\left(\dfrac{x}{x+1}\right)^{x^2} = \lim\limits_{x\to\infty}\left(\dfrac{1}{1+\frac{1}{x}}\right)^{x^2} = \lim\limits_{x\to\infty}\left(\dfrac{1}{\left(1+\frac{1}{x}\right)^x}\right)^x = 0$

⇒ iz limesa „$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$" možemo izvesti neke druge važne limese:

• $t=\frac{1}{x}$, $\begin{matrix}x\to\infty\\ t\to 0\end{matrix}$ $\quad\lim_{t\to 0}(1+t)^{\frac{1}{t}} = e$

$$\boxed{\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e}$$

• $\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{1}{x}\cdot\ln(1+x) = \lim_{x\to 0}\left(\ln(1+x)^{\frac{1}{x}}\right)$

(neprekidna funkcija i limes komutiraju)

$= \ln\left(\lim_{x\to 0}(1+x)^{\frac{1}{x}}\right)$

$= \ln e = 1$

$$\boxed{\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1}$$

• $\lim_{x\to 0}\frac{e^x-1}{x} = \lim_{x\to 0}\frac{t}{\ln(1+t)} = \lim_{x\to 0}\frac{1}{\frac{\ln(1+t)}{t}} = 1$

$e^x-1=t \quad \left(\begin{matrix}x\to 0\\ t\to 0\end{matrix}\right)$

$e^x = t+1$

$x = \ln(t+1)$

$$\boxed{\lim_{x\to 0}\frac{e^x-1}{x} = 1}$$

ZAD: $\lim_{x\to 0^+}\left(\frac{e^{2x}-1}{x}\cdot \operatorname{arctan}\left(\frac{1}{x}\right)\right)=\lim_{x\to 0^+}\left(\frac{e^{2x}-1}{2x}\cdot 2\cdot \operatorname{arctan}\left(\frac{1}{x}\right)\right)=\pi$

($\frac{e^{2x}-1}{2x}\to 1$, $\operatorname{arctan}\left(\frac{1}{x}\right)\to\frac{\pi}{2}$)

ZAD: $\lim_{x\to 0}\left(1+\sin x\right)^{3\operatorname{ctg} x}=\lim_{x\to 0}\left(1+\sin x\right)^{\frac{3\cos x}{\sin x}}=\lim_{x\to 0}\left[\left(1+\sin x\right)^{\frac{1}{\sin x}}\right]^{3\cos x}$

$=e^{\lim_{x\to 0} 3\cos x}=e^{3}$

ZAD: $\lim_{x\to 0}\frac{\sin^3(2x)}{\ln(1+x^3)}=\lim_{x\to 0}\frac{\sin^3(2x)}{(2x)^3}\cdot(2x)^3\cdot\frac{x^3}{\ln(1+x^3)}\cdot\frac{1}{x^3}=8$

($\frac{\sin^3(2x)}{(2x)^3}\to 1$, $\frac{x^3}{\ln(1+x^3)}\to 1$)

ZAD: $\lim_{x\to\infty}\left(\operatorname{arch} x-\ln x\right)=\lim_{x\to\infty}\left(\ln\left(x+\sqrt{x^2-1}\right)-\ln x\right)$

$=\lim_{x\to\infty}\left(\ln\frac{x+\sqrt{x^2-1}}{x}\right)=\lim_{x\to\infty}\ln\frac{1+\sqrt{1-\frac{1}{x^2}}}{1}=\ln 2$

ZAD: $\lim_{x\to\infty}\frac{\ln(\operatorname{ch}(2x))}{x}=$

ZAD: $\lim_{x\to 3}(x-2)^{\frac{x}{x-3}}=$

(hint $t=x-3$

2

e^3

} DZ

NEIZMJERNO MALE VELIČINE

$\lim_{x\to a} f(x)=0, \quad \lim_{x\to a} g(x)=0$

$\lim_{x\to a}\frac{f(x)}{g(x)} = C \in R^+ \Rightarrow$ „f" i „g" su neizmjerno male veličine istog reda

$\rightarrow C=1 \Rightarrow$ ekvivalentne veličine ($f \sim g$)

PR: $\lim_{x\to 0}\frac{\overbrace{x^2+x}^{\sim x}}{\underbrace{e^{2x}-1}_{\sim 2x}} = \lim_{x\to 0}\frac{\frac{x^2+x}{x}\cdot x}{\frac{e^{2x}-1}{2x}\cdot 2x} = \lim_{x\to 0}\frac{x}{2x} = \frac{1}{2}$

- $\lim_{x\to 0}\frac{x^2+x}{x}=1$
- $\lim_{x\to 0}\frac{e^{2x}-1}{2x}=1$

$[\infty \cdot 0]$

PR: $\lim_{x\to\infty}\frac{x^3+4}{\sqrt{x^5+x^2}}\cdot \arctan\left(\frac{1}{\sqrt{x}}\right)$

- $\lim_{x\to\infty}\frac{\arctan\left(\frac{1}{\sqrt{x}}\right)}{\frac{1}{\sqrt{x}}} = \left|\begin{matrix} t=\frac{1}{\sqrt{x}} \\ x\to\infty \\ t\to 0 \end{matrix}\right| = \lim_{t\to 0}\frac{\arctan t}{t} = \left|\begin{matrix} t=\tan l \\ t\to 0 \\ l\to 0 \end{matrix}\right| = \lim_{l\to 0}\frac{l}{\tan l}$

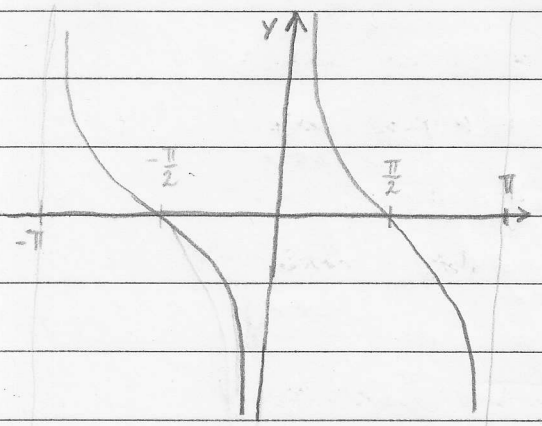
$= \lim_{l\to 0}\frac{l\cos x}{\sin x} = 1 \quad ; \quad \arctan\frac{1}{\sqrt{x}} \sim \frac{1}{\sqrt{x}}, \ x\to\infty$

- $\lim_{x\to\infty}\frac{\frac{x^3+4}{\sqrt{x^5+x^2}}}{\sqrt{x}} = \lim_{x\to\infty}\frac{x^3+4 \,/:x^3}{\sqrt{x^6+x^5}\,/:x^3} = 1 \quad ; \quad \frac{x^3+4}{\sqrt{x^5+x^2}} \sim \sqrt{x}, \ x\to+\infty$

$\lim_{x\to\infty}\left(\sqrt{x}\cdot\frac{1}{\sqrt{x}}\right) = 1$

JEDNOSTRANI LIMESI

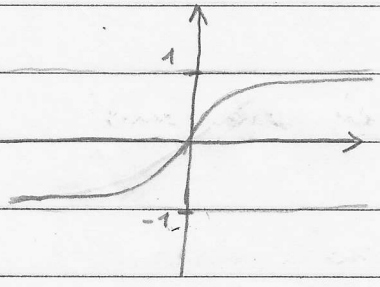
Postoj $\lim_{x \to 0} \operatorname{ctg} x$?



Ne! Ali vidimo da je:

- $\lim_{x \to 0^+} \operatorname{ctg} x = \infty$
- $\lim_{x \to 0^-} \operatorname{ctg} x = -\infty$

Postoj $\lim_{x \to 1} \operatorname{th}\left(\frac{1}{x-1}\right)$?



$\operatorname{th} x$ je rastuća

$\frac{1}{x-1}$ je padajuća

$x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2) \Rightarrow g(f(x_2)) \leq g(f(x$

↳ padajuća funkcija

- $\lim_{x \to 1^-} \operatorname{th}\left(\frac{1}{x-1}\right) = \lim_{\varepsilon \to 0^+} \operatorname{th}\left(\frac{1}{1-\varepsilon-1}\right) = \lim_{\varepsilon \to 0^+} \operatorname{th}\left(\frac{1}{-\varepsilon}\right) = -1$ ($\frac{1}{-\varepsilon} \to -\infty$)

$1-\varepsilon$ (1

$\varepsilon > 0,\ \varepsilon \to 0^+$

- $\lim_{x \to 1^+} \operatorname{th}\left(\frac{1}{x-1}\right) = \lim_{\varepsilon \to 0^+} \operatorname{th}\left(\frac{1}{1+\varepsilon-1}\right) = \lim_{\varepsilon \to 0^+} \operatorname{th}\left(\frac{1}{\varepsilon}\right) = 1$

1 $1+\varepsilon$

$\varepsilon > 0,\ \varepsilon \to 0^+$

Tražen limes ne postoji ☺

⇒ kada postoji limes funkcije u nekoj točki?

↳ postoji ako postoje limesi s lijeva i desna, te su oni jednaki

⇒ kada je neka funkcija neprekidna u točki „a"?

↳ mora postojati „$\lim_{x\to a} f(x)$" (znači da vrijedi sljedeće: $\lim_{x\to a^-} f(x) = \lim_{x\to a^+} f(x) = L$) i on mora biti jednak vrijednosti funkcije u toj točki ($L = f(a)$)

DEF: Neka je „f" definirana na otvorenom intervalu koji sadrži točku „a". Kažemo da je „f" neprekidna u „a" ako je $\lim_{x\to a} f(x) = f(a)$

PR: $f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ ?, & x = 0 \end{cases}$

↳ što treba pisati umjesto „?" da bi funkcija bila neprekidna?

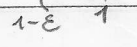
↳ treba pisati 1 ∇° (vidi se iz grafa) :)

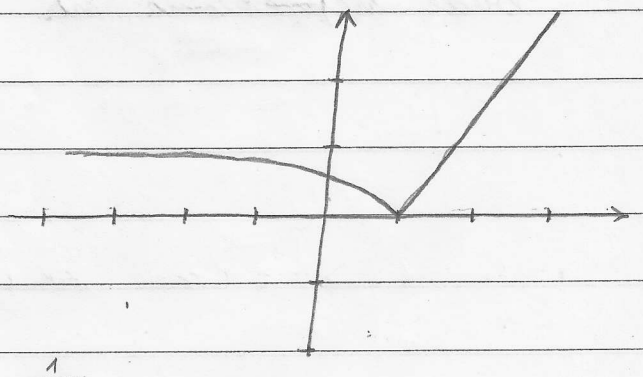
ZAD: Odredi parametar „$a \in R$" tako da funkcija $f(x) = \begin{cases} e^{\frac{1}{x-1}}, & x<1 \\ ax-1, & x \geq 1 \end{cases}$ bude neprekidna na R ∇!

① $f_1(x) = e^{\frac{1}{x-1}}$

• nije definirana za $x=1$

• $\lim_{x\to 1^-} e^{\frac{1}{x-1}} = 0$





$$\lim_{x\to-\infty} e^{\frac{1}{x-1}} = \left|\begin{matrix} t=-x \\ x\to-\infty \\ t\to+\infty \end{matrix}\right| = \lim_{t\to\infty} e^{\frac{1}{-t-1}} = \lim_{t\to\infty} e^{\frac{1}{t+1}} = 1$$

② $f_2(x) = ax-1$

Da bi f bila neprekidna u $x=1$, mora biti $\lim_{x\to 1} f(x) = f(1)$

$0 = a-1 \Rightarrow a=1$

ZAD: Koliki treba biti $a \in R$, tako da $f(x) = \begin{cases} \frac{\sin(ax)}{x}, & x<0 \\ 2e^x, & x \geq 0 \end{cases}$ bude neprekidna na cijelom R?

↳ problem je u $x=0$

$$\lim_{x\to 0} f(x) = f(0)$$

$$\left.\begin{array}{l} \bullet \lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} \frac{\sin(ax)}{a\cdot x}\cdot a = a \\ \bullet \lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} (2e^x) = 2 \end{array}\right\} \text{trebaju biti jednaki, pa: } a = 2$$

ZAD: Odredi „a" i „b" tako da je $f(x) = \begin{cases} a + \arctan\left(\frac{1}{x}\right), & x<0 \\ b, & x=0 \\ \arctan\left(\frac{1}{x}\right), & x>0 \end{cases}$ neprekidna na cijelom R ▽

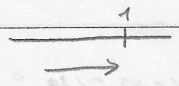
$$\left.\begin{array}{l} \lim\limits_{x\to 0^-} \left(a + \arctan\left(\frac{1}{x}\right)\right) = a - \frac{\pi}{2} \\ \lim\limits_{x\to 0^+} \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2} \end{array}\right\} a - \frac{\pi}{2} = \frac{\pi}{2}$$

$$a = \pi$$

$$\frac{\pi}{2} = f(0) = b \Rightarrow b = \frac{\pi}{2}$$

ZAD: Odredi „a", tako da je $f(x) = \begin{cases} x^2 \sin\left(\frac{1}{x^2}\right), & x \neq 0 \\ a, & x = 0 \end{cases}$ bude neprekidna na R ▽

ZAD: $\lim\limits_{x\to 1^-} \frac{x^2-3x+2}{x^2-2x+1} = \lim\limits_{x\to 1^-} \frac{(x-1)(x-2)}{(x-1)^2} = \lim\limits_{x\to 1^-} \frac{x-2}{x-1} \left[\frac{-1}{0^-}\right] = \infty$



ZAD: $\lim\limits_{x\to 1^+} \frac{x^2-3x+2}{x^2-2x+1} = \ldots = \lim\limits_{x\to 1^+} \frac{x-2}{x-1} \left[\frac{-1}{0^+}\right] = -\infty$

ZAD: $\lim\limits_{x\to -2^-} e^{\frac{x-2}{x+2}} = e^{\lim\limits_{x\to -2} \frac{x-2}{x+2}} = e^{\infty} = \infty$

ZAD: Postoj ℓ limes funkcije $f(x) = cth \frac{3}{x^2-9}$ u točki $x=3$?

$\cdot \lim\limits_{x\to 3^+} cth \frac{3}{x^2-9} = \lim\limits_{x\to 3^+} cth \frac{3}{(x-3)(x+3)} =$

DERIVACIJE FUNKCIJE

⇒ povod za derivacije:

• pojam brzine:

$$\bar{v} = \frac{s(t_0+\Delta t) - s(t_0)}{\Delta t}$$

"NEWTON"

$$v(t_0) = \lim_{\Delta t \to 0} \frac{s(t_0+\Delta t) - s(t_0)}{\Delta t}$$

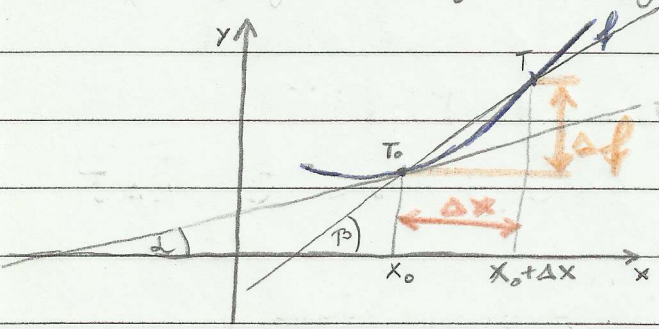
PR: Odredi brzinu tijela koje slobodno pada u $t_0 = 4\,s$, ako je put dan funkcijom $s(t) = \frac{1}{2} g t^2$. ∇

$$v(4) = \lim_{\Delta t \to 0} \frac{s(4+\Delta t) - s(4)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\frac{1}{2} g (4+\Delta t)^2 - \frac{1}{2} g \cdot 4^2}{\Delta t}$$

$$v(4) = \frac{1}{2} g \cdot \lim_{\Delta t \to 0} \frac{16 + 8\Delta t + (\Delta t)^2 - 16}{\Delta t} = \frac{1}{2} g \lim_{\Delta t \to 0} (8 + \Delta t)$$

$$v(4) = \frac{1}{2} g \cdot 8 = 4 \cdot g = 40\ m/s$$

• konstrukcija tangente na krivulji:



"LEIBNITZ"

$$k_t = \operatorname{tg}\alpha = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

tangenta

$$k_s = \operatorname{tg}\beta = \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x}$$

sekanta

PR: Odredi jednadžbu tangente na krivulji $y = x^2$ u točki $T_0(2, 4)$ ∇

$$y - 4 = k_t \cdot (x - 2)$$

$$y - 4 = 4(x + 2$$

⋮

sred !

$$k_t = \lim_{\Delta x \to 0} \frac{(2+\Delta x)^2 - 2^2}{\Delta x}$$

$$k_t = 4$$

DEFINICIJA: $f: (I \subseteq R) \to R$; I je otvoren interval ; $x_0 \in I$

↳ ako postoji broj $f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$, onda taj broj zovemo derivacija funkcije „f" u točki „x_0"

⇒ jednadžba tangente na $G(f)$ u točki $T_0(x_0, y_0)$:

$$y - y_0 = f'(x_0) \cdot (x - x_0)$$

⇒ VAŽNE DERIVACIJE:

a) $f(x) = c \quad c \in R$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0$$

$$\boxed{c' = 0}$$

b) $f(x) = x$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{x+h-x}{h} = 1$$

$$\boxed{x' = 1}$$

c) $f(x) = \frac{1}{\sqrt{x}}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{\sqrt{x+h}} - \frac{1}{\sqrt{x}}}{h} = \lim_{h \to 0} \frac{\frac{\sqrt{x} - \sqrt{x+h}}{\sqrt{x+h} \cdot \sqrt{x}}}{h} = \lim_{h \to 0} \frac{\sqrt{x} - \sqrt{x+h}}{h \cdot \sqrt{x+h} \cdot \sqrt{x}}$$

$$f'(x) = \lim_{h \to 0} \frac{x - (x+h)}{h \cdot \sqrt{x+h} \cdot \sqrt{x} (\sqrt{x} + \sqrt{x+h})} = \frac{-1}{2x\sqrt{x}}$$

$$f'(x) = -\frac{1}{2} x^{-\frac{3}{2}}$$

d) $f(x) = \frac{1}{x}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} = \lim_{h \to 0} \frac{\frac{x - (x+h)}{x(x+h)}}{h} = \lim_{h \to 0} \frac{-1}{x^2 + xh} = -\frac{1}{x^2}$$

↳ opčenito za oblik: $f(x) = x^{\alpha}$; $\alpha \in R$

$$\boxed{f'(x) = \alpha \cdot x^{\alpha-1}}$$

e) $f(x) = \ln x$

$$f'(x) = \lim_{h\to 0} \frac{\ln(x+h) - \ln x}{h} = \lim_{h\to 0} \frac{\ln\left(\frac{x+h}{h}\right)}{h} = \lim_{h\to 0} \frac{\ln\left(1+\frac{x}{h}\right)}{\frac{h}{x}\cdot x}$$

$$f'(x) = \frac{1}{x}$$

$$\boxed{(\ln x)' = \frac{1}{x}}$$

f) $f(x) = \sin x$

$$f'(x) = \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

($\sin x \to 0$, $\frac{\cos h - 1}{h} \to 0$, $\cos x \to \cos x$, $\frac{\sin h}{h} \to 1$)

$$= \cos x$$

g) $f(x) = \cos x$

$$f'(x) = \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} = \lim_{h\to 0} \frac{\cos x \cos h - \sin x \sin h}{h}$$

$$= \lim_{h\to 0} \left(\frac{\cos x \cos h}{h} - \frac{\sin x \sin h}{h} \right)$$

($\frac{\cos h}{h} \to 0$, $\frac{\sin h}{h} \to 1$)

$$= -\sin x$$

ZAD: Odredi jednadžbu tangente na graf funkcije, ako je:

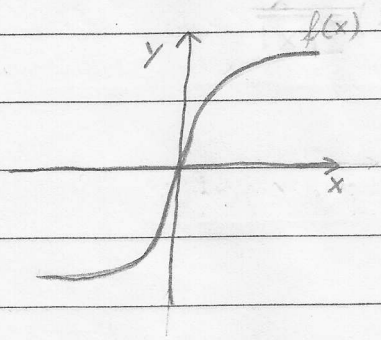
a) $f(x)=\sqrt{x}$, $T(1,1)$

$$f'(x)=\frac{1}{2\sqrt{x}} \Rightarrow f'(1)=\frac{1}{2}$$

$$y-1=f'(1)(x-1)$$

$$y-1=\frac{1}{2}(x-1) \rightarrow y=\frac{1}{2}x+\frac{1}{2}$$

b) $f(x)=\sqrt[3]{x}$, $T(0,0)$



$$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}=\frac{1}{3\cdot\sqrt[3]{x^2}}$$

$$\lim_{x\to 0^+}\frac{1}{3\sqrt[3]{x^2}}=\lim_{x\to 0^-}\frac{1}{3\cdot\sqrt[3]{x^2}}=+\infty$$

↳ jednadžba tangente: $x=0$

c) $f(x)=3+(x-2)^{\frac{1}{3}}$, $T(2,3)$

↳ slična promjena iz „b)"

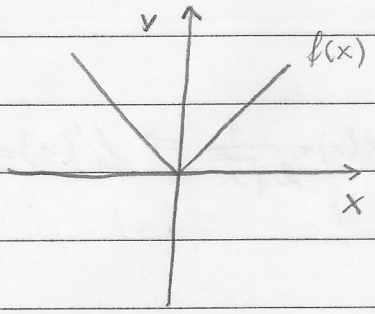
$$f'(x)=\frac{1}{3\cdot\sqrt[3]{(x-2)^2}} \qquad f'(2)=???$$

$$\lim_{x\to 2^+}\frac{1}{3\cdot\sqrt[3]{(x-2)^2}}=\lim_{x\to 2^-}\frac{1}{3\cdot\sqrt[3]{(x-2)^2}}=\infty$$

↳ jednadžba tangente:

$$x=2$$

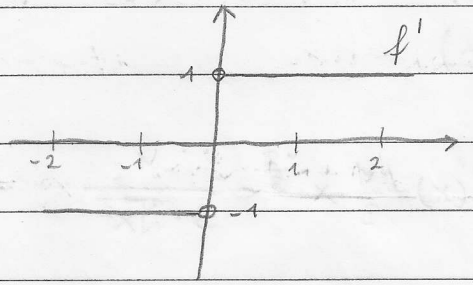
d) $f(x) = |x|$, $T(0,0)$



$$\cdot \lim_{h \to 0^-} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^-} \frac{-h-0}{h} = -1$$

$$\cdot \lim_{h \to 0^+} \frac{f(0+h)-f(0)}{h} = \lim_{h \to 0^+} \frac{h-0}{h} = 1$$

↳ $f'(0)$ ne postoji ! (uoči ŠILJAK)



$$f'(x) = \frac{x}{|x|}$$

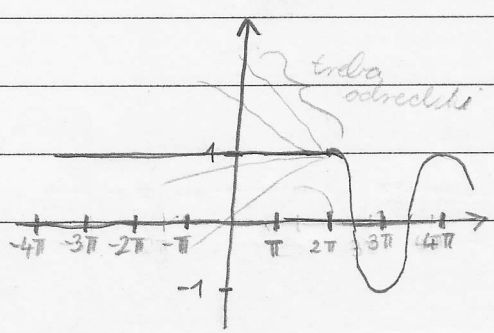
e) $f(x) = |x-1|$, $T(1,0)$

$$f'(x) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0} \frac{|1+h-1|-0}{h} = \lim_{h \to 0} \frac{|h|}{h}$$

↳ $f'(1)$ ne postoji !

ZAD: Odredi parametre „a" i „b" takve da funkcija „f" bude neprekidna i derivabilna u svakoj točki !

$$f(x) = \begin{cases} ax+b, & x < 2\pi \\ \cos x, & x \geq 2\pi \end{cases}$$



• neprekidnost:

$$\lim_{x \to 2\pi^-} (ax+b) = 2\pi \cdot a + b$$

$$\lim_{x \to 2\pi^+} (\cos x) = \cos(2\pi) = 1$$

$$2\pi \cdot a + b = 1$$

• derivabilnost:

$$\left.\begin{array}{l} \lim\limits_{x\to 2\pi^-} a = a \\ \lim\limits_{x\to 2\pi^+} (-\sin x) = 0 \end{array}\right\} \; a = 0 \;\Rightarrow\; b = 1$$

⇒ DERIVACIJA I NEPREKIDNOST:

↳ *teorem:* ako je „f" derivabilna u točki „x", onda je „f" neprekidna u točki „x"

↳ *dokaz:*

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \left\} \text{pretpostavka}\right.$$

↳ treba pokazat neprekidnost u x, a to je kada:

$$\lim_{h\to 0} f(x+h) = f(x)$$

$$\lim_{h\to 0} \left[f(x+h)-f(x) \right] = 0$$

$$\lim_{h\to 0} \underbrace{\frac{f(x+h)-f(x)}{h}}_{\to f'(x)} \cdot \underbrace{h}_{\to 0} = 0$$

$$f'(x)\cdot 0 = 0$$

⇓ pokazali smo:

$$\lim_{h\to 0} f(x+h) = f(x)$$

↳ NAPOMENA: neprekidnost ne povlači diferencijabilnost (npr. $|x|$ u $x=0$)

PRAVILA DERIVIRANJA

⇒ „f" i „g" su derivabilne u „x"

- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$

- $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$

↳ dokaz:

$$(f(x) \cdot g(x))' = \lim_{h \to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x+h) + f(x)g(x+h) - f(x)g(x+h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \left[\frac{f(x+h) - f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h) - g(x)}{h} \right]$$

zbog neprekidnost: $g(x+h) = g(x)$

$$= f'(x) \cdot g(x) + f(x) \cdot g'(x)$$

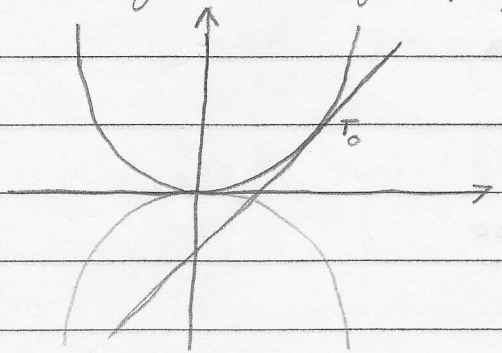
- $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ (dokaz u knjizici)

PR: Odredi derivaciju od ctg X !

$$f(x) = \operatorname{ctg} X = \frac{\cos X}{\sin X}$$

$$f'(x) = \frac{\cos' X \cdot \sin x - \cos x \cdot \sin' X}{\sin^2 x} = \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\frac{1}{\sin^2 x}$$

ZADO: Za koje $a \in R$ je pravac $y = 2x - 6$ tangenta na $y = ax^4$?



$T_0(x_0, y_0)$

$f(x) = a \cdot x^4$

$f'(x) = 4ax^3$

y

$y - y_0 = f'(x_0)(x - x_0)$

$4ax_0^3 = 2$ $\qquad$ $2x_0 - 6 = ax_0^4$

sustav

$a = \frac{2}{4x_0^3} \longrightarrow 2x_0 - 6 = \frac{2}{4x_0^3} \cdot x_0$

$3x_0 = 12 \Rightarrow x_0 = 4$

$a = \frac{2}{4 \cdot 4^3} = \frac{1}{128}$

DERIVACIJA KOMPOZICIJE FUNKCIJA

PR: $h(x) = (x^3+x)^{100}$, $h'(x) = ?$

- $g(x) = x^3+x$, $f(x) = x^{100}$
- $h(x) = (f \circ g)(x)$

$$\boxed{(f \circ g)'(x) = f'(g(x)) \cdot g'(x)}$$ → „ulančano pravilo"

$f'(x) = 100\,x^{99}$; $g'(x) = 3x^2+1$

$$h'(x) = 100\,(x^3+x)^{99} \cdot (3x^2+1)$$

⇒ NEKE OZNAKE:

- $u = g(x)$, $f' = \frac{df}{dx}$

$$\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

PR: $u = x^3+x$, $f(x) = x^{100}$

$$\frac{df(u)}{dx} = \frac{df}{du} \cdot \frac{du}{dx} = \frac{d}{du}(u^{100}) \cdot \frac{d}{dx}(x^3+x)$$

$$= 100\,u^{99} \cdot (3x^3+x)^{99} = 100\,(x^3+x)^{99} \cdot (3x^2+1)$$

PR: $H(x) = \tan^3\left(\frac{\pi x}{3}\right)$

- $h(x) = \frac{\pi x}{3}$, $g(x) = \tan x$, $f(x) = x^3$

$$H'(x) = (f \circ g \circ h)(x)$$

$$H'(x) = (f \circ g \circ h)'(x) = (f \circ g)'(h(x)) \cdot h'(x) =$$
$$= f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$$

$$f'(x) = 3x^2, \quad g'(x) = \frac{1}{\cos x}, \quad h'(x) = \frac{\pi}{3}$$

$$H'(x) = 3\tan^2\left(\frac{\pi x}{3}\right) \cdot \frac{1}{\cos\left(\frac{\pi x}{3}\right)} \cdot \frac{\pi}{3}$$

ZAD: Izvedi derivaciju kvocijenta ∇!

$$\frac{f(x)}{g(x)} = f(x) \cdot (g(x))^{-1}$$

$$\left(\frac{f(x)}{g(x)}\right)' = f'(x) \cdot (g(x))^{-1} + f(x) \cdot (-1)(g(x))^{-2} \cdot g'(x)$$

$$= \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$$

DERIVACIJA INVERZNE FUNKCIJE

$(f \circ f^{-1})(y) = y \;/\; \frac{d}{dy}$

$y = f(x)$

$x = f^{-1}(y)$

$f'(f^{-1}(y)) \cdot (f^{-1})'(y) = 1$ ☐

$$\boxed{(f^{-1})'(y) = \frac{1}{f'(x)}}$$

PR: $f(x) = \sin x = y$; $x = f^{-1}(y) = \arcsin y$ $\left(y \in \langle -1, 1 \rangle \right.$

$x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

$$(\arcsin y)' = \frac{1}{(\sin x)'} = \frac{1}{\cos x} \overset{+}{=} \frac{1}{\sqrt{1-\sin^2 x}}$$

$\cos x > 0$ za $x \in \langle -\frac{\pi}{2}, \frac{\pi}{2} \rangle$

$$= \frac{1}{\sqrt{1-\sin^2(\arcsin y)}} = \frac{1}{\sqrt{1-y^2}}$$

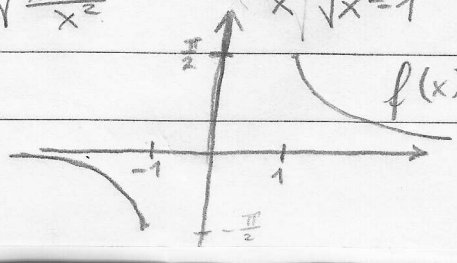
$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

ZAD: Izračunaj $\left(\arcsin\left(\frac{1}{x}\right)\right)' \Big|_{x=-\frac{2}{\sqrt{3}}}$ ▽

$f(x) = \arcsin\left(\frac{1}{x}\right) \longrightarrow -1 \le \frac{1}{x} \le 1 \Rightarrow x \in \langle -\infty, -1] \cup [1, \infty \rangle$

$$f'(x) = \frac{1}{\sqrt{1-\left(\frac{1}{x}\right)^2}} \cdot (-1) \cdot x^{-2} = -\frac{1}{x^2 \cdot \sqrt{\frac{x^2-1}{x^2}}} = -\frac{1}{x\sqrt{x^2-1}} = -\frac{|x|}{x^2\sqrt{x^2-1}}$$

$f'\left(-\frac{2}{\sqrt{3}}\right) = \ldots = -\frac{3}{2}$



PR: $y = f(x) = \cos x \qquad x \in [0, \pi]$

$x = f^{-1}(y) = \arccos y$ $\qquad -\frac{1}{\sqrt{1-x^2}}$

$$\arccos(y) = \frac{1}{(\cos x)'} = -\frac{1}{\sin x} = -\frac{1}{\sqrt{1-\cos^2 x}} = -\frac{1}{\sqrt{1-y^2}}$$

$$\arccos x = -\frac{1}{\sqrt{1-x^2}}$$

PR: $(\operatorname{arctan} x)' = ?$ $\qquad y = f(x) = \tan x$

$x = f^{-1}(y) = \operatorname{arctan} y$

$$(\operatorname{arctan} x)' = \frac{1}{(\tan x)'} = \frac{1}{\frac{1}{\cos^2 x}} = \cos^2 x$$

$$= \frac{1}{1+\tan^2 x} = \frac{1}{1+y^2}$$

$$(\operatorname{arctan} x)' = \frac{1}{1+x^2}$$

PR: $y = f(x) = \ln x \qquad x = f^{-1}(y) = e^y$

$$(e^y)' = \frac{1}{(\ln x)'} = \frac{1}{\frac{1}{x}} = x = e^y$$

$$e^{x'} = e^x$$

PR: $y = f(x) = \operatorname{sh} x$

$x = f^{-1}(y) = \operatorname{arsh} y$

$$(\operatorname{arsh} y)' = \frac{1}{(\operatorname{sh} x)'} = \frac{1}{\operatorname{ch} x} = \frac{1}{\sqrt{1+\operatorname{sh}^2 x}} = \frac{1}{\sqrt{1+y^2}}$$

$$(\operatorname{sh} x)' = \frac{1}{2}(e^x + e^{-x}) = \operatorname{ch} x$$

$$\operatorname{arsh} x = \frac{1}{\sqrt{1+x^2}}$$

⇒ kako deriviramo funkcije oblika:

a) $f(x) = a^x \quad a > 0, \; a \neq 1$

1. NAČIN:

$$f(x) = a^x = e^{\ln a^x} = e^{x \ln a}$$

$$f'(x) = e^{x \ln a} \cdot \ln a = a^x \cdot \ln a$$

2. NAČIN: logaritamsko deriviranje

$$f(x) = a^x \; / \ln$$

$$\ln f(x) = x \ln a \; / \frac{d}{dx}$$

$$\frac{1}{f(x)} \cdot f'(x) = \ln a$$

$$f'(x) = f(x) \cdot \ln a = a^x \cdot \ln a$$

b) $(x^\alpha)' = \alpha \cdot x^{\alpha - 1}, \quad \alpha \in R$

$$f(x) = x^\alpha = e^{\alpha \ln x}$$

$$f'(x) = e^{\alpha \ln x} \cdot \frac{\alpha}{x} = x^\alpha \cdot \frac{\alpha}{x} = \alpha \cdot x^{\alpha - 1}$$

ZAD: $f(x) = \frac{x^3 \cdot e^x}{1 + x^2} \; / \ln$

$$\ln f(x) = 3 \ln x + x - \ln(1 + x^2) \; / \frac{d}{dx}$$

$$\frac{1}{f(x)} \cdot f'(x) = \frac{3}{x} + 1 - \frac{1}{1 + x^2} \cdot 2x$$

$$f'(x) = \frac{x^3 \cdot e^x}{1 + x^2} \left(\frac{3}{x} + 1 - \frac{2x}{1 + x^2} \right)$$

ZAD: $f(x) = (\cos x)^{\sin x}$, $f'(x) = ?$

$f(x) = \cos x$

$$(f(x))^{g(x)} = e^{\ln\left((f(x))^{g(x)}\right)} = e^{g(x)\cdot \ln f(x)}$$

$$f'(x) = e^{\sin x \cdot \cos x} \cdot \left[\cos x \cdot \ln(\cos x) + \sin x \cdot \frac{1}{\cos x} \cdot (-\sin x)\right]$$

$$= (\cos x)^{\sin x} \cdot \left[\cos x \cdot \ln(\cos x) - \frac{\sin^2 x}{\cos x}\right]$$

ZAD: Postoj li $f'(0)$ ako je $f(x) = \sqrt{1-\sqrt{1-x^2}}$?

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{\sqrt{1-\sqrt{1-h^2}}}{h} \cdot \frac{\sqrt{1+\sqrt{1-h^2}}}{\sqrt{1+\sqrt{1-h^2}}} = \lim_{h\to 0} \frac{\sqrt{1-(1-h^2)}}{h\cdot\sqrt{1+\sqrt{1-h^2}}}$$

$$= \lim_{h\to 0} \frac{|h|}{h\cdot\sqrt{1+\sqrt{1-h^2}}}$$

limesi s lijeva i desna su različiti

$$\lim_{h\to 0^+} \frac{|h|}{h\sqrt{1+\sqrt{1-h^2}}} = \lim_{h\to 0^+} \frac{h}{h\cdot\sqrt{1+\sqrt{1-h^2}}} = \frac{1}{\sqrt{2}}$$

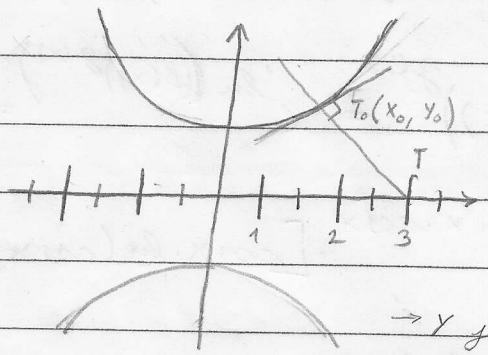
$$\lim_{h\to 0^-} \frac{|h|}{h\sqrt{1+\sqrt{1-h^2}}} = -\frac{1}{\sqrt{2}}$$

↳ limesi su različiti pa derivacija ne postoji !

Zad: Odredi točku na hiperboli $y=\sqrt{x^2+1}$ najbližu točki $T(3,0)$

$y=\sqrt{x^2+1} \quad /^2$

$y^2-x^2=1$



→ y je parabola pa ovo ne gledamo

↳ nađi normalu na $G(f)$ u $T(3,0)$:

$f(x)=\sqrt{x^2-1}$

$f'(x)=\frac{1}{2}(x^2+1)^{-\frac{1}{2}}\cdot 2x=\frac{x}{\sqrt{x^2+1}}$

$y-y_0=-\frac{1}{f'(x_0)}(x-x_0)$

$y-y_0=-\frac{1}{\frac{x_0}{\sqrt{x_0^2+1}}}(x-x_0)$

$y-y_0=-\frac{\sqrt{x_0^2+1}}{x_0}(x-x_0)$

↳ $T(3,0)$ leži na normali:

$0-y_0=-\frac{\sqrt{x_0^2+1}}{x_0}(3-x_0)$

$x_0y_0=(3-x_0)\cdot\sqrt{x_0^2+1}$

↳ $T_0(x_0,y_0)$ leži na krivulji: $y_0=\sqrt{x_0^2+1}$ } sustav

$x_0y_0=(3-x_0)y_0, \quad y_0\neq 0$

$x_0=3-x_0 \Rightarrow x_0=\frac{3}{2}$

$y_0=\sqrt{\frac{9}{4}+1}=\sqrt{\frac{13}{4}}$

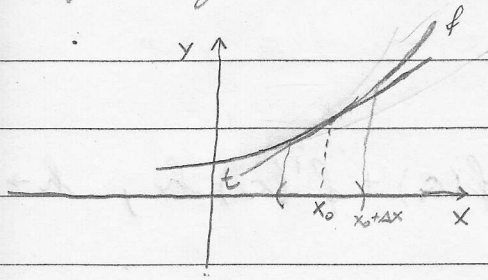
$y_0=\frac{\sqrt{13}}{2}$

$T_0\left(\frac{3}{2},\frac{\sqrt{13}}{2}\right)$

DIFERENCIJAL FUNKCIJE

$y = f(x)$

- $f(x)$ je derivabilna u točki x_0



$$y - f(x_0) = f'(x_0)(x - x_0)$$
$$y = f(x_0) + f'(x_0)(x - x_0)$$

$$f(x_0+\Delta x) \approx f_0(x_0) + f'(x_0)(x_0+\Delta x - x_0)$$
$$f(x_0+\Delta x) - f(x_0) \approx f'(x_0)\cdot \Delta x$$
$$\Delta f \approx f'(x_0)\cdot \Delta x$$

$$\boxed{df = f'(x_0)\,dx} \Rightarrow \text{diferencijal funkcije u točki}$$

↳ NAPOMENA: za neki „x", diferencijal funkcije „f" u točki „x" u oznaci $df(x)$ je linearna funkcija u varijabli „h":

$$(df(x))(h) = f'(x)\cdot h$$

↳ skraćeno pišemo:

$$df(x) = f'(x)\cdot h$$

ZAD: Pomoću diferencijala aproksimiraj vrijednost od $\sqrt{4.1}$

$f(x) = \sqrt{x}$, $x_0 = 4$, $h = 0.1$; $f'(x) = \frac{1}{2\sqrt{x}}$

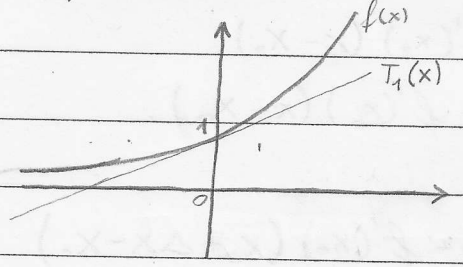
$$\sqrt{4.1} = f(x_0+h) \approx \underbrace{f(x_0) + f'(x_0)\cdot h}_{df(x_0)} = 2 + \frac{1}{2\sqrt{4}}\cdot 0.1 = 2 + \frac{1}{40} = \frac{81}{40}$$

$\sqrt{4.1} \approx 2.025$

TAYLOROVA FORMULA

⇒ možemo li bolje aproksimirat? Npr. ako aproksimiramo polinomom je bolje nego ako aproksimiramo pravcem?

PR: $f(x) = e^x$ u okolini točke $c = 0$



$$f(c+h) \approx f(c) + f'(c)\,h, \quad h \to 0$$

$$c = 0$$

$$e^x \approx 1 + 1 \cdot (x-0) = \underbrace{1+x}_{T_1(x)}$$

$$T_1(0) = 1 = f(0)$$

$$T_1'(0) = f'(0)$$

⇒ DEFINICIJA:

↳ Taylorov polinom n-tog stupnja za funkciju „f" u okolini točke „c"

- $I \subseteq R$ je otvoren interval
- $f: I \to R$ u točki „c" ima sve derivacije $(f', f'', \ldots, f^{(n)})$

$$T_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

ZAD: Za zadanu („dovoljno glatku") funkciju odredi polinom „n"-tog stupnja tako da vrijedi:

$$T_n(c) = f(c),\quad T_n'(c) = f'(c),\ \dots,\ T_n^{(n)}(c) = f^{(n)}(c)$$

↳ opći oblik polinoma: $T_n(x) = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots + a_n(x-c)^n$

$$T_n(c) = f(c) \Rightarrow a_0 = f(c)$$

$$T_n'(x) = a_1 + a_2 \cdot 2(x-c) + \dots + a_n \cdot n(x-c)^{n-1}$$

$$T_n'(c) = f'(c) = a_1 \Rightarrow a_1 = f'(c)$$

$$T_n''(x) = a_2 \cdot 2 + a_3 \cdot 3 \cdot 2(x-c) + \dots + a_n \cdot n(n-1)(x-c)^{n-2}$$

$$T_n''(c) = f''(c) = a_2 \cdot 2 \Rightarrow a_2 = \frac{f''(c)}{2}$$

$$\vdots$$

$$T_n^{(n)}(x) = a_n \cdot n(n-1)(n-2)\cdot\dots\cdot(n-(n-1))(x-c)^{n-n} = a_n \cdot n!$$

$$T_n^{(n)} = f^{(n)}(c) \Rightarrow a_n = \frac{f^{(n)}(c)}{n!}$$

PR: $f(x) = e^x$ u okolini točke $c = 0$! (NASTAVAK ?)

$T_1(x) = 1 + x$

- $T_2(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2$ → T_1

$T_2(x) = 1 + x + \frac{1}{2}x^2$

- $T_3(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^2$ → T_2

$$T_2(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$$

ZAD: Napišite $T_3(x)$ za funkcijo $f(x) = \sin x$ v okolici točke $c = \frac{\pi}{2}$.

$$T_3(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \frac{f'''(c)}{3!}(x-c)^3$$

$$f(x) = \sin x \Rightarrow f'(x) = \cos x \Rightarrow f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$

$$f\left(\tfrac{\pi}{2}\right) = 1 \quad f'\left(\tfrac{\pi}{2}\right) = 0 \quad f''\left(\tfrac{\pi}{2}\right) = -1 \quad f'''\left(\tfrac{\pi}{2}\right) = 0$$

$$T_3(x) = 1 + \frac{0}{1!}\left(x - \tfrac{\pi}{2}\right) + \frac{-1}{2!}\left(x - \tfrac{\pi}{2}\right)^2 + \frac{0}{3!}\left(x - \tfrac{\pi}{2}\right)^3$$

$$T_3(x) = 1 - \frac{1}{2}\left(x - \tfrac{\pi}{2}\right)^2 \rightarrow \text{kvadratna funkcija}$$

↳ stoga, kako nebi bilo zabune, kaže se da imamo Taylorov polinom „n"-tog reda, a ne stupnja!

ZAD: Napiši $T_4(x)$ za funkciju $f(x) = \cos x$ u okolini točke $c = 0$!

$$T_4 = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{IV}(0)}{4!}x^4$$

$$f'(x) = -\sin x \Rightarrow f'(0) = 0$$
$$f''(x) = -\cos x \Rightarrow f''(0) = -1$$
$$f'''(x) = \sin x \Rightarrow f'''(0) = 0$$
$$f^{IV}(x) = \cos x \Rightarrow f^{IV}(0) = 1$$

$$T_4 = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

TOČNOST APROKSIMACIJE

⇒ kolika dobro polinom $T_n(x)$ aproksimira funkciju ?

⇒ TAYLOROV TEOREM:

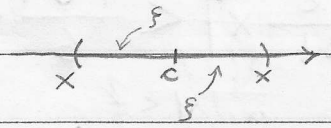
- $f: I \to R$, $I \subseteq R$, $c \in I$
- „f" ima derivacije do reda $f^{(n+1)}$ u $x = c$

↳ tada funkciju „f" možemo prikazat u obliku:

$$f(x) = \underbrace{T_n(x)}_{\text{TAYLOROV POLINOM}} + \underbrace{R_n(x)}_{\text{OSTATAK}}$$

↳ ostatak zapisujemo ovako:

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot (x-c)^{n+1}$$, za ovaj broj „ksi" (ξ) između „x" i „c"



⇒ kolika je dobra aproksimacija funkcije Taylorovim polinomom $T_n(x)$ u okolini točke „c" ?

↳ možemo ocijeniti ostatak $R_n(x)$!

$$f(x) = T_n(x) + R_n(x)$$

$$|f(x) - T_n(x)| = |R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-c)^{n+1} \right| \leq \varepsilon$$

pretpostavljamo

↳ željena točnost

npr. $|f^{(n+1)}(t)| \leq M$, $\forall t$ između x i c

$$|R_n(x)| = \left| \frac{f^{(n+1)}(\xi)}{(n+1)!} (x-c)^{n+1} \right| \leq M \cdot \frac{|(x-c)^{n+1}|}{(n+1)!} \; ; \; \lim_{n \to \infty} R_n(x) = 0$$

↳ no, ne možemo uvijek „n" poslati u beskonačno

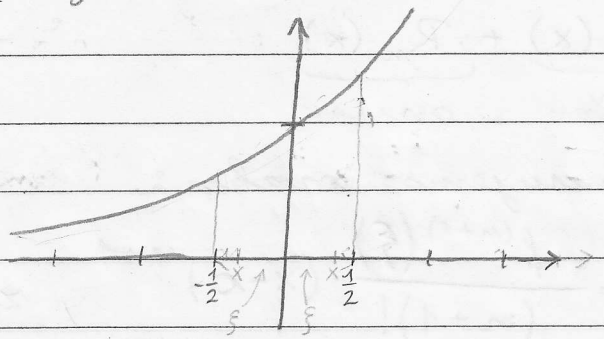
PR: Ocejni pogrešku aproksimacije funkcije $f(x) = e^x$ Taylorovim polinomom $T_2(x) = 1 + x + \frac{x^2}{2}$ na intervalu $I = (-\frac{1}{2}, \frac{1}{2})$ ∀.

$$f(x) = T_2(x) + R_2(x)$$

$$T_2(x) = 1 + x + \frac{x^2}{2}, \quad R_2(x) = \frac{f^{(3)}(\xi)}{3!}(x-c)^3,$$

$$R_2(x) = \frac{e^\xi}{3!} \cdot x^3$$

↳ ocenimo sada R_2 za $x \in \langle -\frac{1}{2}, \frac{1}{2} \rangle$:



$$|R_2(x)| = \left|\frac{e^\xi}{3!} \cdot x^3\right| < \frac{|e^\xi|}{6} \cdot \frac{1}{8} = \frac{e^\xi}{48} < \frac{e^{\frac{1}{2}}}{48} \approx 0.034$$

↳ $|x| < \frac{1}{2}$ ↳ ocjena pogreške

$x < 0$: $\xi \in (x, 0)$ pa $e^\xi < e^0 < e^{\frac{1}{2}}$

$x > 0$: $\xi \in (0, x)$ pa $e^\xi < e^{\frac{1}{2}}$

↳ aproksimiraj: npr. $e^{\frac{1}{4}} \approx 1 + \frac{1}{4} + \frac{1}{16 \cdot 2} = 1.28125$

↳ digitron: 1.2840254

$$\left|f\left(\frac{1}{4}\right) - T_2\left(\frac{1}{4}\right)\right| = 0.0027754$$

↳ stvarna pogreška (uvijek manja od ocjene pogreške)

⇒ napomena:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad , \quad \forall x \in R$$

$$\lim_{n \to \infty} R_n(x) = 0$$

ZAD: $f(x) = \ln\left(\frac{1+x}{1-x}\right)$, prikaži u obliku $f(x) = T_3(x) + R_3(x)$, $c = 0$?

$$f(x) = T_3(x) + R_3(x) = f(x) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(\xi)}{4!}x^4 \quad , \quad \xi \text{ između } 0 \text{ i } x$$

$$f(x) = \ln(1+x) - \ln(1-x)$$

$$f'(x) = \frac{1}{1+x} - \frac{1}{1-x} \cdot (-1) = \frac{1}{1+x} + \frac{1}{1-x} = (1+x)^{-1} + (1-x)^{-1}$$

$$f'(x) = (-1)(1+x)^{-2} + (-1)(1-x)^{-2}$$

$$\vdots$$

$$f(0) = 0 \quad , \quad f'(0) = 2 \quad , \quad f''(0) = 0 \quad , \quad f'''(0) = 4$$

$$f^{IV}(x) = -6(1+x)^{-4} + 6(1-x)^{-4}$$

$$f(x) = T_3(x) + R_3(x) = 0 + \frac{2}{1!}x + \frac{0}{2!}x^2 + \frac{4}{3!}x^3 + (-6)\left(\frac{1}{(1+\xi)^4} - \frac{1}{(1-\xi)^4}\right)\frac{x^4}{4!}$$

$$= 2x + \frac{2}{3}x^3 + (-6)\left(\frac{1}{(1+\xi)^4} - \frac{1}{(1-\xi)^4}\right)\frac{x^4}{4!}$$

L'HOSPITALOVO PRAVILO

⇒ služi za računanje limesa oblika $\frac{0}{0}$ i $\frac{\infty}{\infty}$

↳ ostali se svode na jedan od ta dva oblika

- $\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = 0$

„ili"

$\lim\limits_{x \to x_0} f(x) = \lim\limits_{x \to x_0} g(x) = \infty$

- postoj $\lim\limits_{x \to x_0} \frac{f'(x)}{g'(x)} \in \overline{R}$, $\overline{R} = R \cup \{-\infty\} \cup \{\infty\}$,

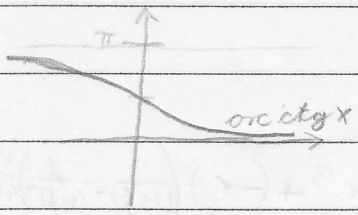
onda je:

$$\boxed{\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}}$$

Pr: $\lim\limits_{x \to \infty} x \cdot \operatorname{arc\,ctg}(2x) = ?$ $\quad (\infty \cdot 0)$

↳ druge mogućnosti:

$\frac{x}{\frac{1}{\operatorname{arcctg}(x+2)}}$ ili $\frac{\operatorname{arc\,ctg}(2x)}{\frac{1}{x}}$



→ lakše ☺

$$\lim_{x \to \infty} x \cdot \operatorname{arcctg}(2x) = \lim_{x \to \infty} \frac{\operatorname{arc\,ctg}(2x)}{\frac{1}{x}} \overset{\frac{0}{0}}{=} \lim_{x \to \infty} \frac{-\frac{1}{4x^2} \cdot 2}{-\frac{1}{x^2}} = \frac{1}{2}$$

↳ napomena: $\boxed{(\operatorname{arc\,ctg} x)' = -\frac{1}{1+x^2}}$

↳ dokaži!

Pr: $\lim\limits_{x\to 0}\left(\frac{1}{\sin x}-\frac{1}{x}\right) \overset{\infty-\infty}{=} \lim\limits_{x\to 0}\left(\frac{x-\sin x}{x\sin x}\right) \overset{\frac{0}{0}}{=} \lim\limits_{x\to 0}\left(\frac{1-\cos x}{1\cdot\sin x+x\cos x}\right) =$

$$= \lim_{x\to 0}\frac{1-\cos x}{\sin x+x\cos x} \overset{\frac{0}{0}}{=} \lim_{x\to 0}\frac{-(-\sin x)}{\cos x+1\cdot\cos x+x\cdot(-\sin x)} =$$

$$= \lim_{x\to 0}\frac{\sin x}{2\cos x-x\sin x} = \frac{0}{2} = 0$$

Zad: U ovisnosti o $n\in\mathbb{N}$, odredi limes:

$$\lim_{x\to 0^+}\frac{1-\cos x-\frac{\tan^2 x}{2}}{x^n} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x\to 0^+}\frac{\sin x-\frac{1}{2}\cdot 2\tan x\cdot\frac{1}{\cos^2 x}}{n\cdot x^{n-1}}$$

$$= \lim_{x\to 0^+}\frac{\sin x-\frac{\sin x}{\cos^3 x}}{n\cdot x^{n-1}} \overset{\left[\frac{0}{0}\right]}{=} \lim_{x\to 0^+}\frac{\sin x\left(1-\frac{1}{\cos^3 x}\right)}{n\cdot x^{n-1}} = \lim_{x\to 0^+}\frac{\sin x\,(\cos^3 x-1)}{n\cdot x^{n-1}\cdot\cos^3 x}$$

$$= \lim_{x\to 0^+}\frac{\sin x(\cos x-1)\overbrace{(\cos^2 x+\cos x+1)}^{\to 3}}{n\cdot x^{n-1}\cdot\underbrace{\cos^3 x}_{\to 1}}$$

$$= \lim_{x\to 0^+}\left[\frac{1}{n}\cdot\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\frac{(\cos x-1)(\cos^2 x+\cos x+1)}{\cos^3 x}}_{\to 3}\cdot\frac{1}{x^{n-2}}\right]$$

$$\frac{\cos x-1}{x^{n-2}} = -\underbrace{\frac{2\sin^2\left(\frac{x}{2}\right)}{\frac{x^2}{4}\cdot 4}}_{\to 1}\cdot\frac{1}{x^{n-4}}$$

$$= \lim_{x\to 0^+}\left[\frac{1}{n}\cdot\underbrace{\frac{\sin x}{x}}_{\to 1}\cdot\underbrace{\frac{(-2)\sin^2\left(\frac{x}{2}\right)}{\frac{x^2}{4}\cdot 4}}_{\to -2}\cdot\frac{1}{x^{n-4}}\cdot\underbrace{\frac{\cos^2 x+\cos x+1}{\cos^3 x}}_{\to 3}\right]$$

$$= -\frac{3}{2n}\cdot\lim_{x\to 0^+}\frac{1}{x^{n-4}} = \begin{cases} 0, & n<4 \\ -\frac{3}{8}, & n=4 \\ -\infty, & n>4 \end{cases}$$

Zad: $\lim\limits_{x\to 0^+}\frac{x\cos x+a\sin x}{x} \overset{\left[\frac{0}{0}\right]}{=} \lim\limits_{x\to 0^+}\frac{\cos x+x\cdot(-\sin x)+a\sin x}{3x^2} \overset{\left[\frac{1+a}{0}\right]}{=}$

↳ za $a=-1$, $\lim\limits_{x\to 0^+}\frac{-x\sin x}{3x^2} = -\frac{1}{3}\lim\limits_{x\to 0^+}\frac{\sin x}{x} = -\frac{1}{3}$

↳ za $1+a>0 \Rightarrow a>-1$, ∞

↳ za $a<-1$, $-\infty$

ZAD: $\lim\limits_{x\to 0} \frac{\tan^2(3x)}{x\ln(1+2x)}$ → probaj L'Hospitalovo pravilo, i vidjet ćeš da imaš komplicirani izraz od početnog, pa je bolje probati nešto drugo!

$$\lim_{x\to 0} \frac{\tan^2(3x)}{x\ln(1+2x)} = \lim_{x\to 0} \frac{\frac{\sin^2(3x)}{\cos^2(3x)}}{x\cdot\ln(1+2x)} = \lim \frac{\sin^2(3x)}{(3x)^2}\cdot(3x)^2\cdot\frac{1}{\cos^2(3x)}$$

$$= \lim_{x\to 0} \frac{\sin^2(3x)}{(3x)^2}\cdot(3x)^2\cdot\frac{1}{\cos^2(3x)}\cdot\frac{1}{x\cdot\frac{\ln(1+2x)}{2x}\cdot 2x}$$

$$= \lim_{x\to 0} \frac{9x^2}{2x^2} = \frac{9}{2}$$

ZAD: $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{x\ln x} = e^{\lim\limits_{x\to 0^+}(x\ln x)} = 1$

$$\lim_{x\to 0^+} x\ln x \overset{[0\cdot(-\infty)]}{=} \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{\left[\frac{-\infty}{\infty}\right]}{=} \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} =$$

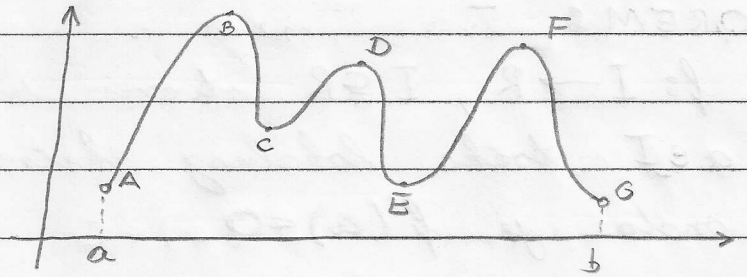
$$= -\lim_{x\to 0^+} x = 0$$

OSNOVNI TEOREMI DIFERENCIJALNOG RAČUNA

⇒ koriste za ispitivanje toka funkcije

⇒ EKSTREMI FUNKCIJE:

$f: [a, b] \to R$



↳ A, C, E, G su lokalni minimumi

↳ A, G su rubne točke

↳ B, D, F su lokalni maksimumi

↳ globalni maksimum: B

↳ globalni minimum: G

PR: $f(x) = \frac{1}{\cos x}$, $x \in [-3\pi, 3\pi]$

↳ „f" nije definirana za $\cos x = 0$; $x = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$

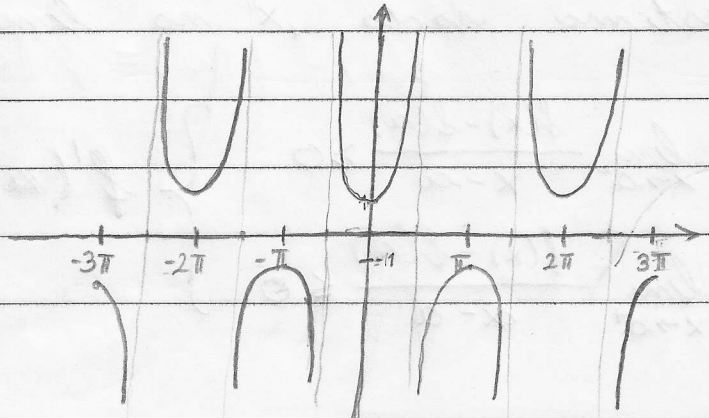
↳ $|f(x)| \geq 1$ jer je $|\cos x| \leq 1$

↳ $f(-x) = \frac{1}{\cos(-x)} = f(x)$ ⇒ parna funkcija, pa crtamo samo na intervalu $[0, 3\pi]$

↳ $f(x + 2\pi) = \frac{1}{\cos(x+2\pi)} = \frac{1}{\cos x} = \frac{1}{\cos x}$ ⇒ „f" je periodična, a period (temeljni) je 2π

↳ $\lim_{x \to \frac{\pi}{2}^-} = \infty$

↳ $\lim_{x \to \frac{\pi}{2}^+} = +\infty$



⇒ primjet:

↳ lokalni minimumi su veći od lokalnih maksimuma

↳ lokalni maksimumi: $-3\pi, -\pi, \pi, 3\pi$

↳ lokalni minimumi: $-2\pi, 0, 2\pi$

↳ nemaju globalnih ekstrema

- $f'(x) = -1 \cdot (\cos x)^{-2} \cdot (-\sin x) = \frac{\sin x}{\cos^2 x}$

  ↳ ako je $f'(x) = 0$, onda je to ekstrem

  ↳ $\sin x = 0$, $x = k\pi$, $k \in \mathbb{Z}$

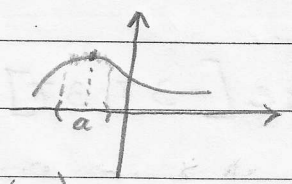
⇒ FERMATOV TEOREM:

↳ neka je $f: I \to \mathbb{R}$, $I \subseteq \mathbb{R}$ otvoren interval

↳ ako je $a \in I$ točka lokalnog ekstrema funkcije „f", onda je $f'(a) = 0$

- DOKAZ:

↳ neka je $a \in I$ točka lokalnog maksimuma $f(x) \leq f(a)$, $\forall x$ iz neke okolne točke „a"



$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

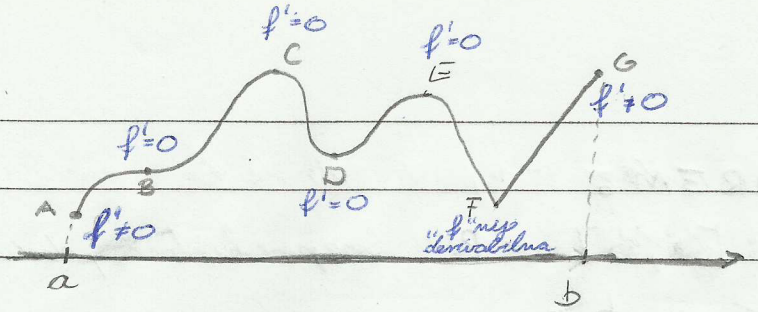
↳ $x \in O(a)$, $x < a$: $\frac{f(x) - f(a)}{x - a} \geq 0$

↳ $x \in O(a)$, $x > a$: $\frac{f(x) - f(a)}{x - a} \leq 0$

⇒ pustimo sada „x" na limes:

$$\left.\begin{aligned} \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a} &\geq 0 \\ \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a} &\leq 0 \end{aligned}\right\} f'(a) = 0$$

PR:



⇒ A, D, F su lokalni minimumi

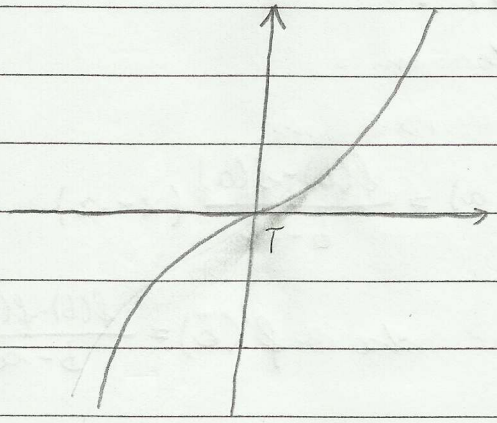
⇒ F je lokalni ekstrem iako „f" tamo nije derivabilna

⇒ B nije lokalni ekstrem iako je „f" tamo derivabilna

⇒ C, E, G su lokalni maksimumi

↳ zaključak: posebno ispitaj rubne točke ∇

PR: $f(x) = x^3$



⇒ PAZI!! T(0,0) nije lokalni ekstrem

⇒ STACIONARNE TOČKE:

- „kandidati za ekstreme", no ne zaboravi rubove segmenta i šiljke funkcije
- stacionarne točke su točke u kojima je prva derivacija funkcije jednaka nul

↳ neka je $f:[a,b]\to R$ neprekidna, derivabilna na intervalu $\langle a,b\rangle$

↳ ako je „$f(a)=f(b)$", onda $\exists c\in\langle a,b\rangle$ tako da je $f'(c)=0$

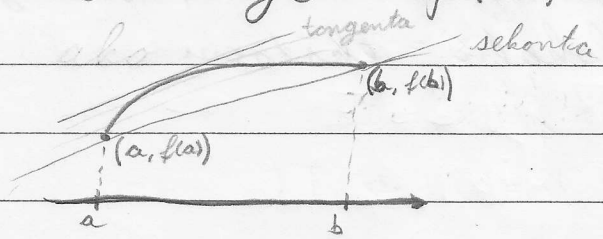
• DOKAZ: a) $f=const$ na $[a,b]$

$\Rightarrow f'(x)=0,\ \forall x\in\langle a,b\rangle$

b) $f\neq const$, onda postoji $c\in\langle a,b\rangle$ koja je minimum ili maksimum od f i $f(c)\neq f(a)$ (a time i $f(b)$)

↳ Fermatov teorem: $f'(c)=0$

⇒ Što ako je $f(a)\neq f(b)$?



$$y-f(a)=\frac{f(b)-f(a)}{b-a}(x-a)$$

↳ $\exists c\in\langle a,b\rangle$ takav da $f'(c)=\frac{f(b)-f(a)}{b-a}$

⇒ LAGRANGEOV TEOREM:

↳ „teorem srednje vrijednosti"

↳ neka je $f:[a,b]\to R$ neprekidna i derivabilna na $\langle a,b\rangle$

↳ tada $\exists c\langle a,b\rangle$ tako da:

$$\boxed{f(b)-f(a)=f'(c)(b-a)}$$

↳ prilično biton teorem, pa prouči ovaj dokaz

• DOKAZ:

↳ definiramo pomoćnu funkciju:

$$F(x) = f(x) - \lambda x \quad , \quad \lambda \in R$$

- odaberemo $\lambda \in R$ tako da $F(a) = F(b)$

jer želimo primjeniti Rolleov teorem

$$F(a) = \underbrace{f(a) - \lambda a = f(b) - \lambda b}_{} = F(b)$$

$$\lambda = \frac{f(b) - f(a)}{b - a} \quad *$$

$\Rightarrow$ „F" ispunjava uvjete Rolleovog teorema, pa možemo zaključiti:

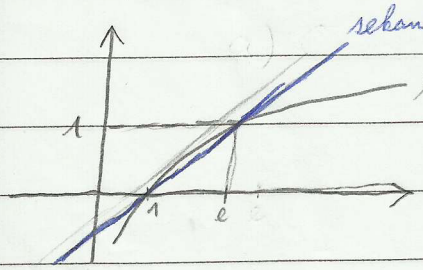
$$\exists c \in \langle a, b \rangle \text{ takav da } F'(c) = 0$$

$$F(x) = f(x) - \lambda x$$

$$F'(x) = f'(x) - \lambda$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

ZAD: Naći točku na krivulji $y = \ln x$, $x \in [1, e]$ u kojoj je tangenta paralelna pravcu koji prolazi točkama krivulje s apscisama $x = 1$, $x = e$.



$$y - 0 = \frac{1 - 0}{e - 1}(x - 1) \quad \rightarrow \text{sekanta}$$

↳ primjena Lagrangeovog teorema:

$$\exists c \in \langle 1, e \rangle \text{ takav da } f'(c) = \frac{1}{e - 1}$$

$$\frac{1}{c} = \frac{1}{e - 1}$$

$$c = e - 1$$

⇒ Kako pomoću prve derivacije odrediti gdje funkcija raste, a gdje pada?

⇒ TEOREM:

↳ neka je I otvoren interval

↳ ako je $f'(x) > 0$, $\forall x \in I$, onda je „f" strogo rastuća na I

↳ ako je $f'(x) < 0$, $\forall x \in I$, onda je „f" strogo padajuća na I

• DOKAZ:

- neka je $f'(x) > 0$, $\forall x \in I$

- odaberimo $x_1, x_2 \in I$ tako da $x_1 < x_2$; primjenimo Lagrangeov teorem:

$$\exists c \in \langle x_1, x_2 \rangle \text{ takav da}$$

$$f(x_2) - f(x_1) = \underbrace{f'(c)}_{>0} \underbrace{(x_2 - x_1)}_{>0}$$

$$\Downarrow$$

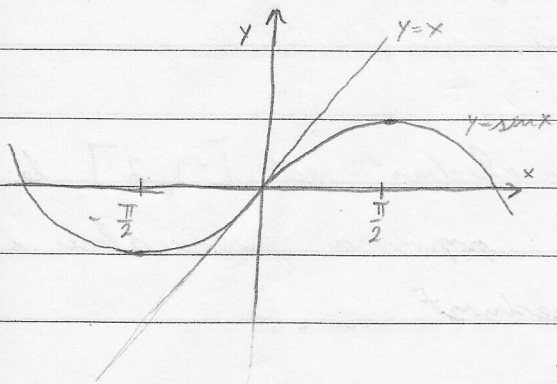
„f" strogo rastuća na I

↳ vrijedi, obrat:

- ako je $f'(x) > 0$, $\forall \in I$, onda je „f" strogo rastuća na I

- naravno, to vrijedi i za padajući interval

ZAD: Pokažite da vrijedi $\sin x < x$, $x > 0$!



$\Rightarrow$ gledamo $x \in \langle 0, \frac{\pi}{2} \rangle$

$\sin 0 = 0$

↳ definiramo s pomoćnu funkciju:

$F(x) = x - \sin x$

$F(0) = 0$

↳ treba pokazat

da je „$F(x)$" rastuća na $\langle 0, \frac{\pi}{2} \rangle$

$F(x) = x - \sin x \overset{?}{>} F(0) = 0$, $x \in \langle 0, \frac{\pi}{2} \rangle$ ← treba pokazat

$F'(x) = 1 - \cos x \overset{?}{>} 0$, $x \in \langle 0, \frac{\pi}{2} \rangle$

$0 < \cos x < 1$

↳ vrijedi za svaki $x \in \langle 0, \frac{\pi}{2} \rangle$

↳ jesu li $\dots, 4\pi, 2\pi, 0, 2\pi, 4\pi, \dots$ ekstremi $F(x)$?

$F'(x) > 0$, $\forall x \in R$, $x \neq 2k\pi$, $k \in Z$

↳ $F(x)$ je strogo rastuća, pa te

točke nisu ekstremi

ZAD: Odredi najveću i najmanju vrijednost funkcije

$f(x) = 3\sqrt[3]{x^2} - 2x \quad , \quad x \in [-1, 2]$

$D(f) = \mathbb{R}$, „f" je neprekidna na $[-1, 2]$ te u tom segmentu poprima minimalnu i maksimalnu vrijednost

↳ kandidati za ekstreme:

- stacionarne točke
- rubovi segmenta
- tama gdje f' nije definirana

↳ $f'(x) = 3 \cdot \frac{2}{3} x^{-\frac{1}{3}} - 2 = \frac{2}{\sqrt[3]{x}} - 2$; $\quad D(f') = \mathbb{R} \setminus \{0\}$

$\frac{2}{\sqrt[3]{x}} - 2 = 0 \quad \Rightarrow \sqrt[3]{x} = 1 \quad \Rightarrow x = 1$

↳ $y = 1$

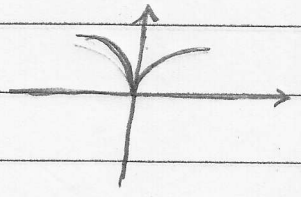
$T(1, 1)$ → stacionarna točka

↳ $f'(x)$ nije definirana za $x = 0$:

$\lim\limits_{x \to 0^-} f'(x) = \lim\limits_{x \to 0^-} \left(\frac{2}{\sqrt[3]{x}} - 2\right) = -\infty$

$\lim\limits_{x \to 0^+} f'(x) = \lim\limits_{x \to 0^+} \left(\frac{2}{\sqrt[3]{x}} - 2\right) = \infty$

} zaključak: u $x = 0$ funkcija ima šiljak:



↳ interval monotonosti:

a) $f'(x) = \frac{2}{\sqrt[3]{x}} - 2 > 0$

$\frac{2 - 2 \cdot \sqrt[3]{x}}{\sqrt[3]{x}} > 0$

$2 - 2 \cdot \sqrt[3]{x} > 0 \quad / : 2$

$\sqrt[3]{x} < 1$

$\sqrt[3]{x} > 0$

$x \in \langle 0, 1 \rangle$

↳ interval rasta

ili $2 - 2 \cdot \sqrt[3]{x} < 0 \quad / : 2$

$\sqrt[3]{x} > 1$

$\sqrt[3]{x} < 0$

$\emptyset$

b) $f'(x) = \frac{2}{\sqrt[3]{x}} - 2 < 0$

$\vdots$

$x \in \langle -\infty, 0 \rangle \cup \langle 1, \infty \rangle$

↳ interval pada

⇒ izradimo sada tablicu:

	$[-1, 0\rangle$	$\langle 0, 1\rangle$	$\langle 1, 2]$
f'	−	+	−
f	↘	↗	↘

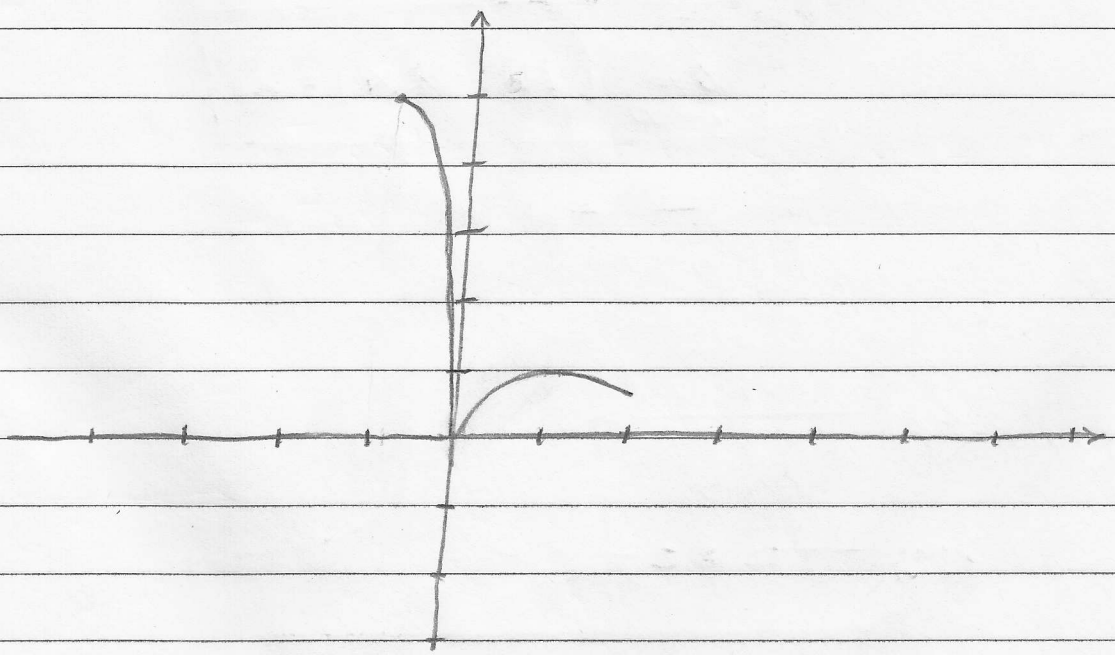
LOKALNI MINIMUM — LOKALNI MAKSIMUM

↳ ispitajmo rubne točke:

$x = -1$, $f(-1) = 3 \cdot 1 + 2 = 5$

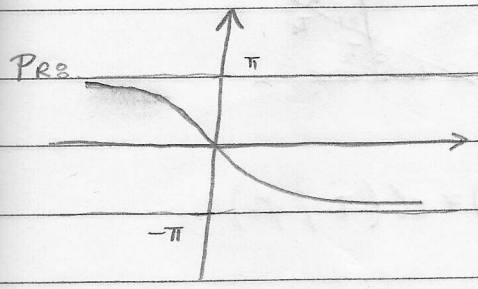
$x = 2$, $f(2) = 3 \cdot \sqrt[3]{4} - 4 \approx 0.7622$

⇒ napokon, nacrtajmo funkciju:



ASIMPTOTE

⇒ već znamo odrediti vertikalne i horizontalne, ali kako znati je l pravac $y = kx + l$ asimptota?



Asimptote: $y = \pi$; $y = -\pi$

PRS $y = \frac{2x+1}{x-1}$ → funkcija nije definirana za $x = 1$, pa je to kandidat za vertikalnu asimptotu

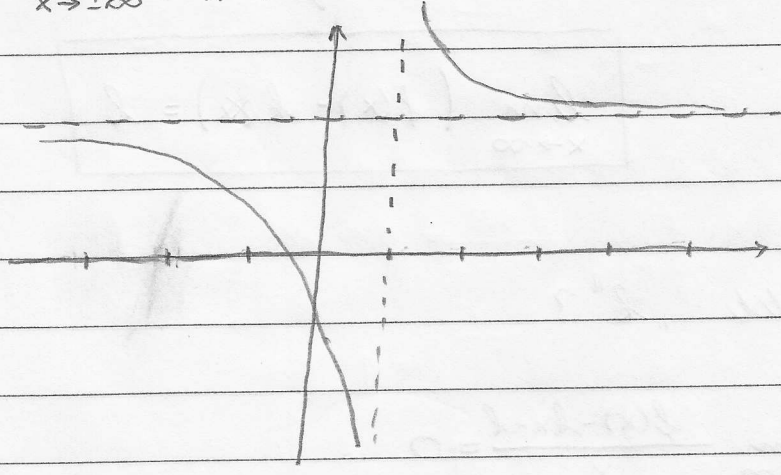
$$\lim_{x \to 1^-} \frac{2x+1}{x-1} = -\infty$$

$$\lim_{x \to 1^+} \frac{2x+1}{x-1} = \infty$$

} $x = 1$ je vertikalna asimptota

↳ imamo li horizontalne asimptote?

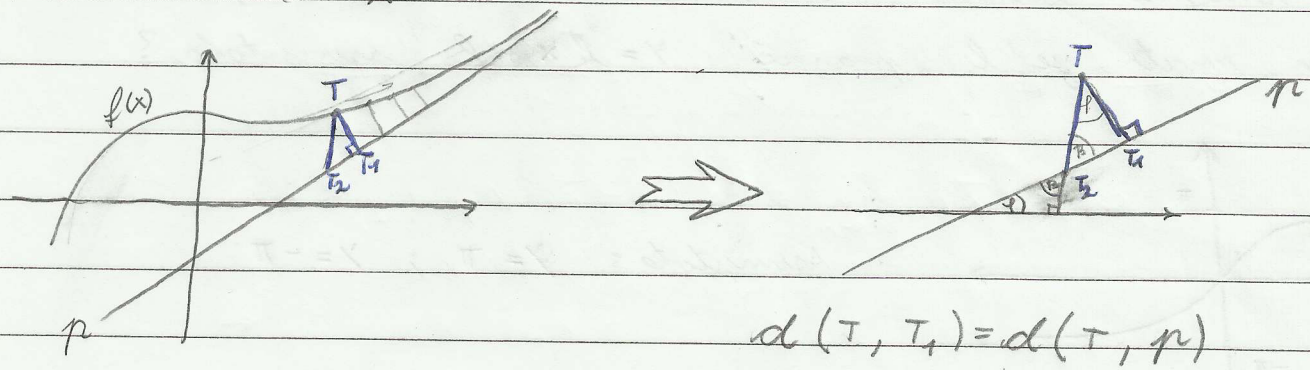
$$\lim_{x \to \pm\infty} \frac{2x+1}{x-1} = 2$$ } $y = 2$ je horizontalna asimptota



↳ primjeti da se udaljenost između funkcije i asimptote smanjuje

⇒ kako tražit kose asimptote funkcije „f"?

↳ npr. pretpostavimo da funkcija ima kosu asimptotu kada $x \to \infty$



$$d(T, T_1) = d(T, p)$$

$d(T, T_1) \to 0$ za $x \to \infty$.

$d(T, T_2) \to 0$ za $x \to \infty$

$$\lim_{x \to \infty} | f(x) - (kx + l) | = 0$$

$$\lim_{x \to \infty} (f(x) - kx - l) = 0$$

↳ dobivamo sljedeće:

$$\boxed{\lim_{x \to \infty} (f(x) - kx) = l}$$

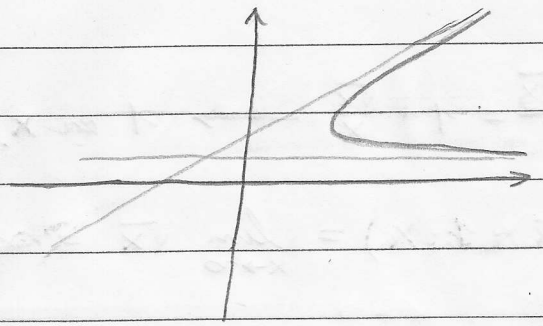
⇒ kako odredit „k"?

$$\lim_{x \to \infty} \frac{f(x) - kx - l}{x} = 0$$

$$\lim_{x \to \infty} \left(\frac{f(x)}{x} - k - \frac{l}{x} \right) = 0 \quad \left(\frac{l}{x} \to 0\right)$$

$$\boxed{k = \lim_{x \to \infty} \frac{f(x)}{x}}$$

⇒ na istoj strani ne mogu biti horizontalna i kosa asimptota jer to više nije funkcija



Zad: Nađi sve asimptote funkcije $y = x \cdot \operatorname{arctan}^2 x$?

$$k = \lim_{x \to \pm\infty} \frac{x \cdot \operatorname{arctan}^2 x}{x} = \lim_{x \to \pm\infty} \left(\operatorname{arctan}^2 x\right) = \left(\pm \frac{\pi}{2}\right)^2 = \frac{\pi^2}{4}$$

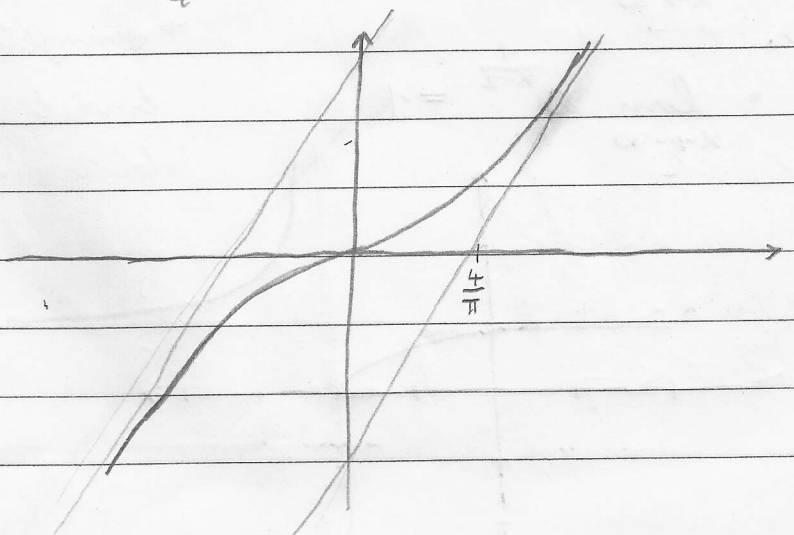
$$l = \lim_{x \to \pm\infty} \left(x \cdot \operatorname{arctan}^2 x - \frac{\pi^2}{4} \cdot x\right) = \lim_{x \to \pm\infty} x\left(\operatorname{arctan}^2 x - \frac{\pi^2}{4}\right) \overset{[\infty \cdot 0]}{=}$$

$$l = \lim_{x \to \pm\infty} \frac{\operatorname{arctan}^2 x - \frac{\pi^2}{4}}{\frac{1}{x}} \overset{\left[\frac{0}{0}\right]}{\underset{L'H}{=}} \lim_{x \to \pm\infty} \frac{2 \operatorname{arctan} x \cdot \frac{1}{1+x^2}}{-\frac{1}{x^2}} =$$

$$= -2 \cdot \lim_{x \to \pm\infty} \left(\operatorname{arctan} x \cdot \frac{x^2}{1+x^2}\right) = \begin{cases} \pi, & x \to -\infty \\ -\pi, & x \to \infty \end{cases}$$

↳ kose asimptote:

- $y = \frac{\pi^2}{4} x - \pi$ (desna kosa asimptota)
- $y = \frac{\pi^2}{4} x + \pi$ (lijeva kosa asimptota)



PR: Promatraj funkciju $f(x) = x + \sqrt{x}$ i nađi njene asimptote ∇

$x \to \infty$

$$\frac{f(x)}{x} = \frac{x+\sqrt{x}}{x} = 1 + \frac{\sqrt{x}}{x} \longrightarrow 1 \text{ za } x \to \infty$$

$$l = \lim_{x\to\infty} \left(x + \sqrt{x} - 1 \cdot x \right) = \lim_{x\to\infty} \sqrt{x} = \infty$$

↳ prinjet da iako je „k" konačan ($k = 1$), „l" nije u beskonačno te funkcija nema asimptota

ZAD: Odredi asimptote funkcije $f(x) = e^{\frac{1}{x-2}}$ i skiciraj graf funkcije ∇

$$D(f) = \mathbb{R} \setminus \{2\}$$

↳ ispitujemo točku $x = 2$:

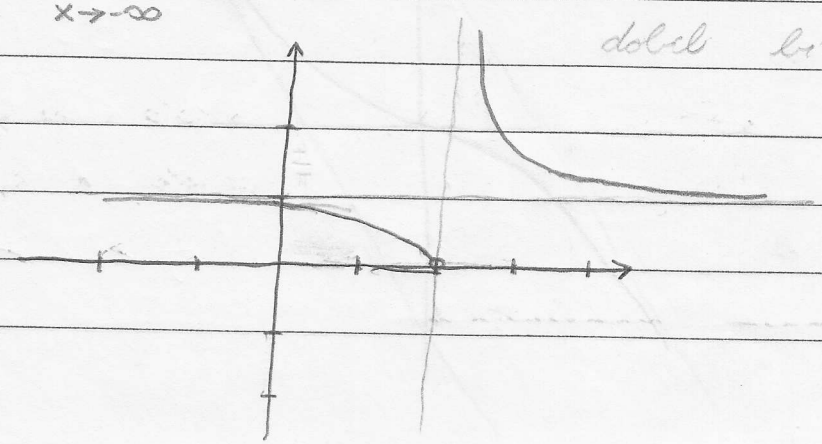
- $\lim\limits_{x\to 2^-} e^{\frac{1}{x-2}} = \left[e^{-\infty} \right] = 0$
- $\lim\limits_{x\to 2^+} e^{\frac{1}{x-2}} = \left[e^{\infty} \right] = \infty$

} $x = 2$ je desna vertikalna asimptota

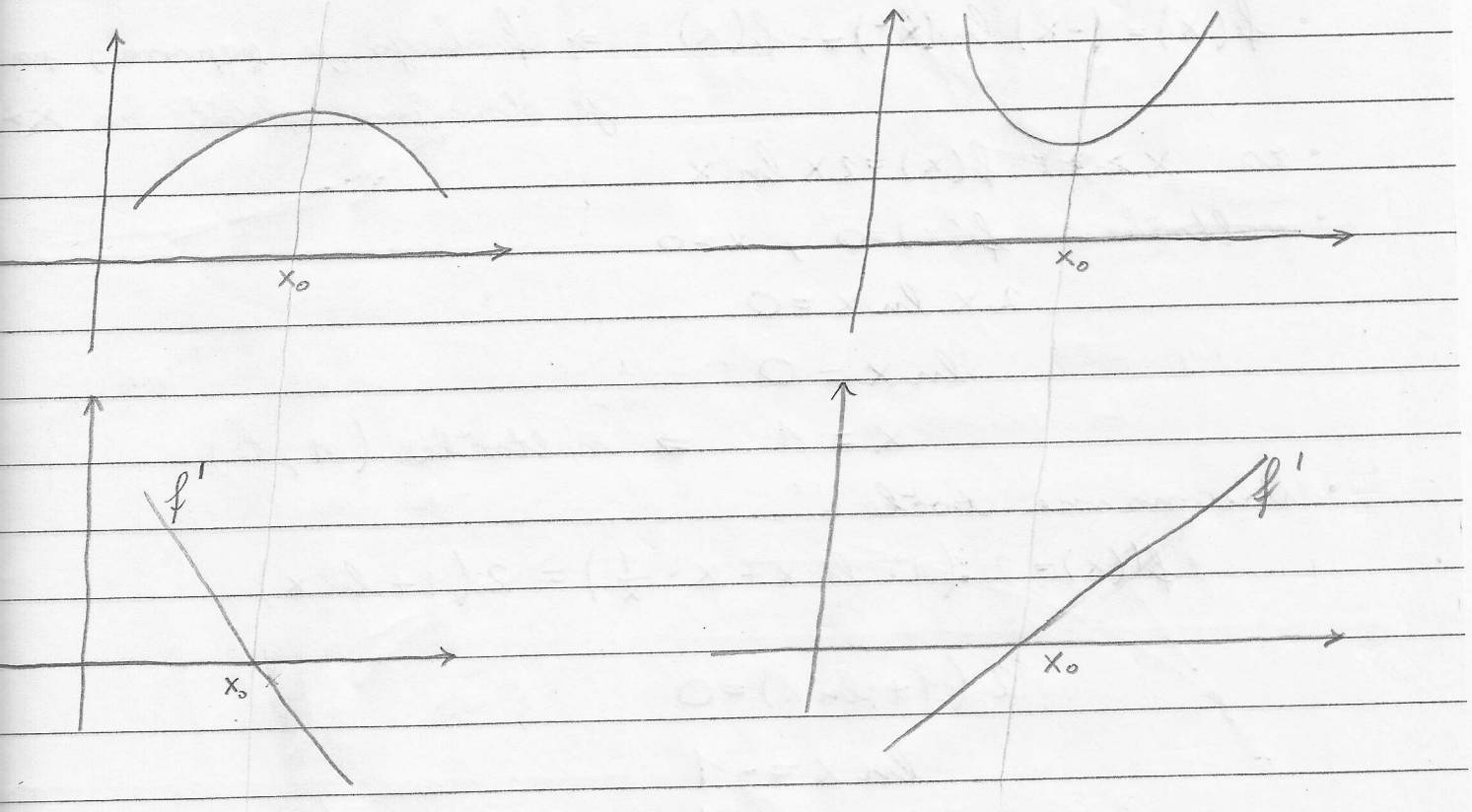
↳ horizontalne asimptote:

- $\lim\limits_{x\to\infty} e^{\frac{1}{x-2}} = 1$
- $\lim\limits_{x\to-\infty} e^{\frac{1}{x-2}} = 1$

⇒ prinjet da smo išli tražit kose asimptote, dobili bi $k = 0$



ODREĐIVANJE KARAKTERA EKSTREMA POMOĆU DRUGE DERIVACIJE



⇒ ZAKLJUČAK ⇒ prolaskom kroz točku lokalnog maksimuma f' pada, pa je $f''(x_0) < 0$

⇒ prolaskom kroz točku lokalnog minimuma, f' raste, pa je $f''(x_0) > 0$

↳ „test" za karakterizaciju ekstrema:

- neka je „f" dvaput derivabilna na intervalu oko točke x_0, te neka je $f'(x_0) = 0$

a) ako je $f''(x_0) > 0$, onda je x_0 lokalni minimum

b) ako je $f''(x_0) < 0$, onda je x_0 lokalni maksimum

c) ako je $f''(x_0) = 0$, onda ne možemo donijeti nikakav zaključak

ZAD: $f(x) = x \cdot \ln(x^2)$? Ispitaj tijek funkcije !

- $D(f) = R \setminus \{0\}$
- $f(-x) = (-x)\ln(x^2) = -f(x)$ ⇒ funkcija je neparna, pa ju dovoljno gledati za $x > 0$
- za $x > 0$: $f(x) = 2x \ln x$
- nultočke: $f(x) = 0$, $x > 0$

$$2x \ln x = 0$$
$$\ln x = 0$$
$$x = 1 \Rightarrow \text{nultočka } (1, 0)$$

- stacionarne točke:
  - $f'(x) = 2\left(1 \cdot \ln x + x \cdot \frac{1}{x}\right) = 2(1 + \ln x)$

$$2(1 + \ln x) = 0$$
$$\ln x = -1$$
$$x = e^{-1} \Rightarrow x_0 = \frac{1}{e}$$

- intervali monotonosti:

a) $f'(x) > 0$

$$1 + \ln x > 0$$
$$\ln x > -1$$
$$\ln x > \ln \frac{1}{e} \quad \Rightarrow \text{NAPOMENA: „ln x" je rastuća funkcija}$$
$$x > \frac{1}{e}$$

b) $f'(x) < 0$

$$1 + \ln x < 0$$
$$\ln x < \ln \frac{1}{e}$$
$$0 < x < \frac{1}{e}$$

	$(0, \frac{1}{e})$	$(\frac{1}{e}, \infty)$
f'	$-$	$\times$
f	$\searrow$	$\nearrow$

lokalni minimum

- ispitivanje karaktera ekstrema pomoću 2. derivacije (jako nije potrebno):

$$f''(x) = \frac{2}{x}$$

$$f''\left(\frac{1}{e}\right) = \frac{2}{\frac{1}{e}} = 2e \Rightarrow 2e > 0$$

↳ „f'' u x_0 ima lokalni minimum"

- asimptote:

a) $$\lim_{x \to 0^+} (2x \ln x) = 2 \lim_{x \to 0^+} (x \cdot \ln x) = 2 \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} \overset{[L'H]}{=}$$

$$= 2 \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = 0$$

↳ njen asimptota jer funkcija ide u nulu

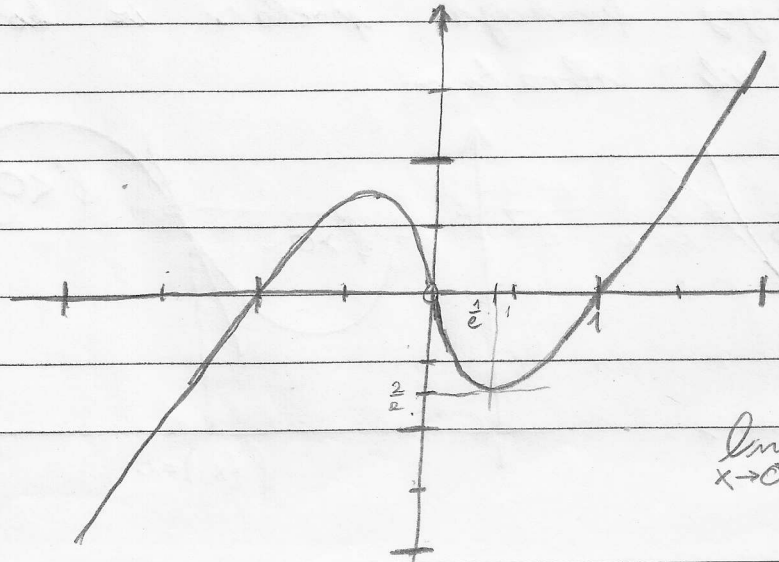
↳ funkcija nema vertikalnih asimptota

b) $$\lim_{x \to \infty} (2x \ln x) = \infty$$

↳ nema horizontalnih asimptota

c) $$k = \lim_{x \to \infty} \frac{f(x)}{x} = \lim_{x \to \infty} \frac{2x \ln x}{x} = \infty$$

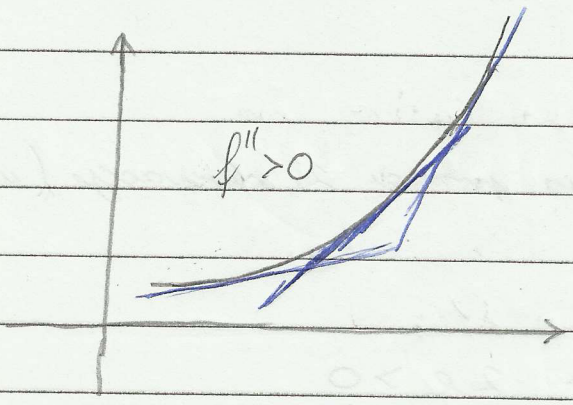
↳ nema kosih asimptota

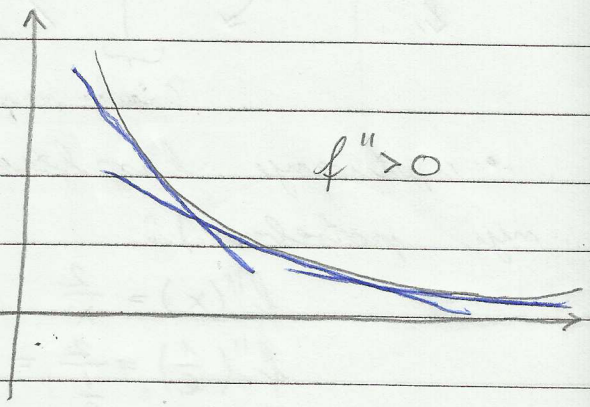


↳ nismo siguni kako funkcija izgleda kod nule, pa ćemo za pomoć koristiti prvu derivaciju

$$\lim_{x \to 0^+} f'(x) = \lim_{x \to 0^+} 2(1 + \ln x) = -\infty$$

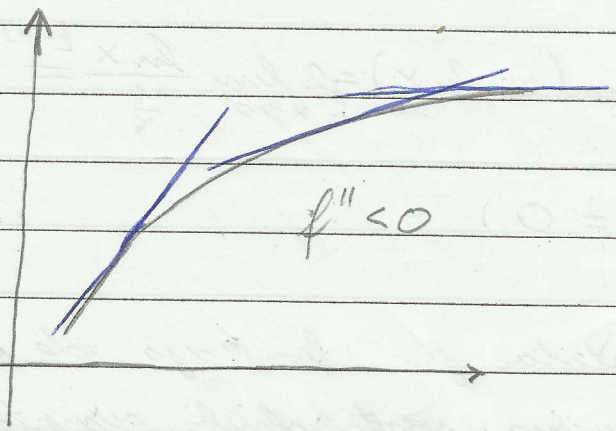
KONVEKSNOST I KONKAVNOST FUNKCIJE

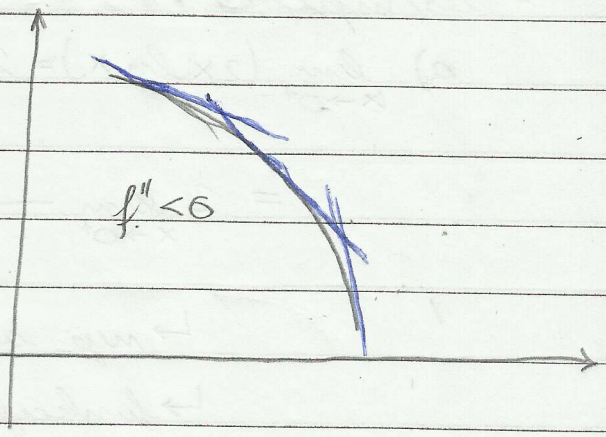




↳ to su konveksne funkcije

↳ tangente su ispod grafa funkcije



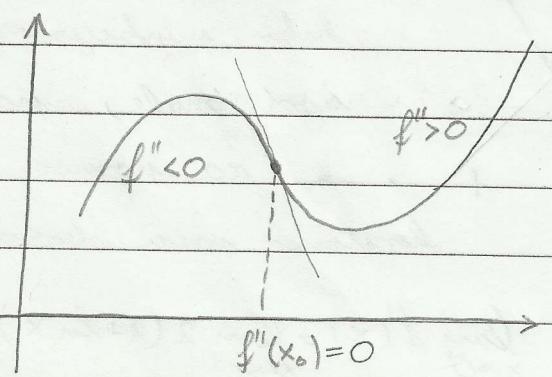


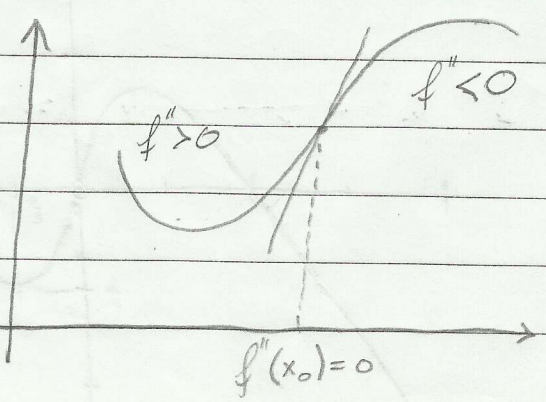
↳ to su konkavne funkcije

↳ tangente su iznad grafa

⇒ TOČKA PREGIBA (INFLEKCIJE):

↳ točka u kojoj funkcija prelazi iz konveksne u konkavnu ili obratno

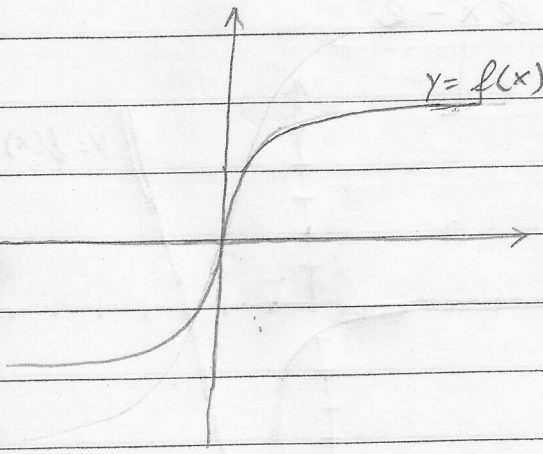




⇒ ako funkcija ima infleksiju u x_0 i ako $f''(x_0)$ postoji, onda je $f''(x_0)=0$ // (nužan uvjet za infleksiju)

⇒ neka je „f" dva puta derivabilna na nekoj okolini od x_0, osim možda u x_0. Ako f'' mijenja predznak u x_0, onda „f" ima infleksiju u x_0. (dovoljan uvjet za infleksiju)

PR: Ispitaj konveksnost i konkavnost funkcije $f(x)=\sqrt[3]{x}$, te odredi točke infleksije



$$f'(x)=\frac{1}{3\sqrt[3]{x^2}}$$

$$f''(x)=\frac{1}{3}\cdot\left(-\frac{2}{3}\right)\cdot x^{-\frac{5}{3}}=-\frac{2}{3}\cdot\frac{1}{\sqrt[3]{x^5}}$$

$f''(x)<0$ na $\langle 0,\infty\rangle$ ⇒ KONKAVNA

$f''(x)>0$ na $\langle -\infty,0\rangle$ ⇒ KONVEKSNA

↳ za $x_0=0$ imamo točku infleksije $(T(0,0)$

↳ uoči da f'' nije definirana $x_0=0$

⇒ KAKO TRAŽIMO TOČKE INFLEKSIJE? $(x_0, f(x)) \in G(f)$

1) odredi sve točke u kojima je $f''(x)=0$ ili f'' nije definirana

2) u svakoj od tih točaka gledamo da li f'' mijenja predznak

ZAD: Skiciraj graf funkcije i odredi jednadžbu tangente u točki infleksije funkcije $f(x) = (x-3)e^x$

$$f'(x) = 1 \cdot e^x + (x-3)e^x = (x-2)e^x$$
$$f''(x) = 1 \cdot e^x + (x-2)e^x = (x-1)e^x$$

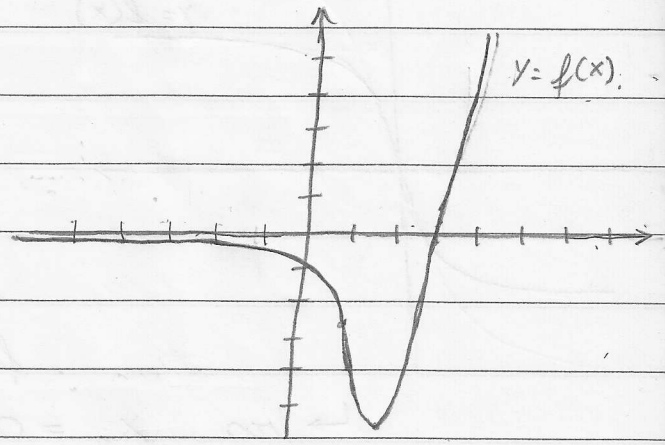
$$f''(x) = 0$$
$$x_0 = 1 \quad \Rightarrow \quad T(1, -2e)$$

$T_0(1, -2e)$

$y - y_0 = f'(x_0)(x - x_0)$

$y - (-2e) = -e(x-1) \quad \Rightarrow \quad y = -ex - e$

	$\langle -\infty, 1\rangle$	$\langle 1, 2\rangle$	$\langle 2, \infty\rangle$
f'	−	−	+
f''	−	+	+
f	↘ ∩	↘ ∪	↗ ∪
		točka infleksije	lokalni minimum



$T_0(1, -2e) \quad M(2, -e^2) \quad N(3, 0)$

↳ asimptote:

- $\lim\limits_{x \to \infty} (x-3)e^x = \infty$

- $\lim\limits_{x \to -\infty} (x-3)e^x = \lim\limits_{x \to \infty} \frac{x-3}{e^{-x}} \overset{L'H}{=} \lim\limits_{x \to -\infty} \frac{1}{-e^{-x}} = -\lim\limits_{x \to -\infty} e^x = 0$

↳ $y = 0$ je lijeva horizontalna asimptota, pa znamo da nema kosih

ISPITIVANJE TOKA FUNKCIJE

ZAD: $f(x) = 1 - x e^{-\frac{2}{x}}$ ⇒ ispitaj tok funkcije i skiciraj ju !

- $D(f) = \mathbb{R} \setminus \{0\}$
- nema parnost ni neparnost, niti periodičnost
- asimptote:

a) $\lim\limits_{x \to 0^+} (1 - x e^{-\frac{2}{x}}) = 1$

$x e^{-\frac{2}{x}} : [0 \cdot e^{-\frac{2}{0^+}}] = 0$

$\lim\limits_{x \to 0^-} (1 - \lim\limits_{x \to 0^-} (x e^{-\frac{2}{x}}) = \lim\limits_{x \to 0^-} \frac{e^{-\frac{2}{x}}}{\frac{1}{x}} \overset{[LH]}{=} \lim\limits_{x \to 0^-} \frac{e^{-\frac{2}{x}}(-2) \cdot \frac{-1}{x^2}}{-\frac{1}{x^2}} = -\infty$

$x e^{-\frac{2}{x}} : [0 \cdot e^{-\frac{2}{0^-}} = 0 \cdot \infty]$

$\lim\limits_{x \to 0^-} (1 - x e^{-\frac{2}{x}}) = 1 - (-\infty) = \infty$

b) $\lim\limits_{x \to \infty} (1 - x e^{-\frac{2}{x}}) = 1 - (\infty) = -\infty$

$x e^{-\frac{2}{x}} : [\infty \cdot e^{-\frac{2}{\infty}} = \infty \cdot e^{0^-} = \infty]$

$\lim\limits_{x \to -\infty} (1 - x e^{-\frac{2}{x}}) = \infty$

↳ nema horizontalnih asimptota

c) kose asimptote:

$k = \lim\limits_{x \to \infty} \frac{f(x)}{x} = \lim\limits_{x \to \infty} \frac{1 - x e^{-\frac{2}{x}}}{x} = \lim\limits_{x \to \infty} \left(\frac{1}{x} - e^{-\frac{2}{x}}\right) = -1$

$l = \lim\limits_{x \to \infty} (f(x) - (-1) \cdot x) = \lim\limits_{x \to \infty} (1 - x e^{-\frac{2}{x}} + x) = \lim\limits_{x \to \infty} (1 + x(1 - e^{-\frac{2}{x}})$

$= 1 + \lim\limits_{x \to \infty} \left(\frac{1 - e^{-\frac{2}{x}}}{\frac{1}{x}}\right) = 1 - \lim\limits_{x \to \infty} \frac{-e^{-\frac{2}{x}}(-2) \cdot \frac{-1}{x^2}}{-\frac{1}{x^2}} = 3$

↳ desna kosa asimptota : $y = -x + 3$

↳ pokaži da je lijeva kosa asimptota $y = -x + 3$

• monotonost :

$$f'(x) = -\left(1 \cdot e^{-\frac{2}{x}} + x \cdot e^{-\frac{2}{x}} \cdot (-2) \cdot \frac{-1}{x^2}\right)$$

$$= -e^{-\frac{2}{x}} - \frac{2}{x} \cdot e^{-\frac{2}{x}} = -e^{-\frac{2}{x}} \cdot \left(1 + \frac{2}{x}\right) = -e^{-\frac{2}{x}} \cdot \frac{x+2}{x}$$

$f'(x) = 0 \Rightarrow x = -2$

$f'(x) > 0$ ili $f'(x) < 0$

$\frac{x+2}{x} < 0$ $\frac{x+2}{x} > 0$

	$\langle -\infty, -2\rangle$	$\langle -2, 0\rangle$	$\langle 0, \infty\rangle$
$x+2$	−	+	+
x	−	−	+
$\frac{x+2}{x}$	+	−	+

$\Downarrow f(x)$

	$\langle -\infty, -2\rangle$	$\langle -2, 0$	$\langle 0, \infty\rangle$
f'	−	+	−
f	↘	↗	↘

lokalni minimum — NIJE DEFINIRANO

$$m(-2, 1 + 2e)$$

• nultočka :

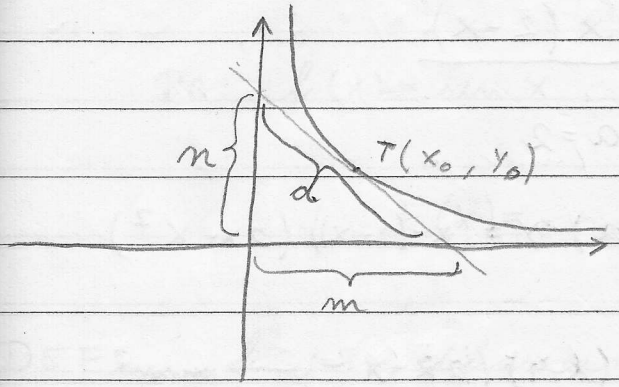
$f(x) = 0$

$1 = x \cdot e^{-\frac{2}{x}}$

$$e^{\frac{2}{x}} = x \Rightarrow \frac{2}{x} = \ln x$$

PRIMJENA EKSTREMA

ZAD: U kojoj točki T krivulje $y = \frac{1}{x^2}$, $x > 0$, treba postaviti tangentu da bi duljina odsječka te tangente između koordinatnih osi bila minimalna?



$$y - y_0 = f'(x_0)(x - x_0)$$

$$f'(x) = \frac{-2}{x^3}$$

$$y - \frac{1}{x_0^2} = -\frac{2}{x_0^3} \cdot (x - x_0)$$

$$y - \frac{1}{x_0^2} = -\frac{2x}{x_0^3} + \frac{2}{x_0^2}$$

$$d = \sqrt{m^2 + n^2}$$

$$d^2 = m^2 + n^2 \to \min$$

odred segmenta oblik

$$\frac{x}{m} + \frac{y}{n} = 1$$

$$\frac{2x}{x_0^3} + y = \frac{3}{x_0^2} \quad / \cdot \frac{x_0^2}{3}$$

$$\frac{2x}{x_0^3} \cdot \frac{x_0^3}{3} + \frac{x_0^2 \cdot y}{3} = 1$$

$$\frac{x}{\frac{3x_0}{2}} + \frac{y}{\frac{3}{x_0^2}} = 1$$

$\frac{3x_0}{2} \to m$, $\frac{3}{x_0^2} \to n$

$$d^2 = \frac{9x_0^2}{4} + \frac{9}{x_0^4}$$

$$F(x) = \frac{9x^2}{4} + \frac{9}{x^4} \to \min \text{ na } \langle 0, \infty \rangle$$

$$F'(x) = \ldots = \frac{9x}{2} - \frac{36}{x^5} = 0$$

$$F'(x) = 0 \Rightarrow x = \sqrt{2}$$

$$F''(x) = \frac{9}{2} + \frac{180}{x^6}; \quad F''(\sqrt{2}) > 0$$

$\hookrightarrow x = \sqrt{2}$ je lokalni minimum

$$T\left(\sqrt{2}, \frac{1}{2}\right)$$

$H(x) = F(x) - G(x)$

$H'(x) = F'(x) - G'(x) = f(x) - f(x) = 0 \quad \forall x \in I$

↳ proizlazi: $H(x) = c, \quad \forall x \in I$

$F(x) - G(x) = c \Rightarrow F(x) = G(x) + c; \quad c \in R$

DEF: Skup svih primitivnih funkcija od „f" na intervalu $\langle a, b \rangle$ zove se neodređeni integral od „f":

⇒ OZNAKA: $\boxed{\int f(x)\, dx = \{ F(x) + C : C \in R}$

PR: Integriraj $f(x) = \frac{1}{x}$

$F(x) = \ln x + C$

↳ pazimo na domene, pa gledaj:

„f" na $\underbrace{\langle -\infty, 0 \rangle}_{I_1} \cup \underbrace{\langle 0, \infty \rangle}_{I_2}$

- $\langle -\infty, 0 \rangle$: $G(x) = \ln(-x) + C$
- $\langle 0, \infty \rangle$: $G(x) = \ln x + C$

↳ dakle:

$F(x) = \ln|x| + C, \quad C \in R$

$$\boxed{\int \frac{dx}{x} = \ln|x| + C} \quad C \in R$$

PR: Integriraj $f(x) = x^3$!

$F'(x) = f(x)$ $\qquad (x^4)' = 4x^3$

$$\int x^3 \, dx = \frac{1}{4} x^4 + C \qquad C \in R$$

↳ splošno:

$$\int x^\alpha \, dx = \frac{x^{\alpha+1}}{\alpha+1} + C \qquad C \in R,\ \alpha \in R \setminus \{-1\}$$

PR: Integriraj $f(x) = \frac{1}{\sqrt{1-x^2}}$

$$\int \frac{dx}{\sqrt{1-x^2}} = \arcsin x + C \qquad C \in R,\ x \in \langle -1, 1 \rangle$$

↳ splošno:

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\left(\frac{x}{a}\right) + C \qquad C \in R,\ a > 0 \to \text{zakaj?}$$

$$\left(\arcsin\left(\frac{x}{a}\right)\right)' = \frac{1}{\sqrt{1-\left(\frac{x}{a}\right)^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{\frac{a^2-x^2}{a^2}}} \cdot \frac{1}{a} =$$

$$= \frac{|a|}{\sqrt{a^2-x^2}} \cdot \frac{1}{a} = \frac{1}{\sqrt{a^2-x^2}} \cdot \frac{1}{\frac{a}{|a|}}$$

tu bi se javio problem

⇒ TEOREM: Linearnost integriranja (BITNO)

„f" i „g" definirane na I i integrabilne

↳ VRIJEDI:

$$(i)\ \int (f(x)+g(x))\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$(ii)\ \int \lambda \cdot f(x)\,dx = \lambda \cdot \int f(x)\,dx\ ;\quad \lambda \in \mathbb{R}$$

↳ primjer:

$$\boxed{T(\lambda \cdot f + \beta \cdot g) = \lambda \cdot T(f) + \beta \cdot T(g)}$$

$\lambda, \beta \in \mathbb{R}$

ZAD: Izračunaj:

$$a)\ \int \frac{\sqrt{x}+1}{x}\,dx = \int (x^{-\frac{1}{2}} + x^{-1})\,dx = \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + \ln|x| + C =$$

$$= 2\sqrt{x} + \ln|x| + C\ ;\quad C \in \mathbb{R}$$

$$b)\ \int \frac{x^2}{x^2+3}\,dx = \int \frac{x^2+3-3}{x^2+3} = \int dx - 3\int \frac{dx}{x^2+3} = x - 3\cdot\frac{1}{\sqrt{3}}\cdot \operatorname{arctg}\left(\frac{x}{\sqrt{3}}\right) + C$$

$$\boxed{\int \frac{dx}{x^2+a} = \frac{1}{a}\operatorname{arctg}\left(\frac{x}{a}\right) + C}$$

$$c)\ \int \operatorname{ctg}^2 x\,dx = \int \frac{\cos^2 x}{\sin^2 x}\,dx = \int \frac{1-\sin^2 x}{\sin^2 x}\,dx =$$

$$= \int \frac{dx}{\sin x} - \int dx =$$

$$\boxed{\int \frac{dx}{\sin^2 x} = -\operatorname{ctg} x + C}$$

ZAD: Izračunaj: (substitucija)

$$\int \sqrt[3]{2x-1}\,dx = \left| \begin{matrix} t = 2x-1 \\ dt = 2\,dx \\ dx = \frac{1}{2}dt \end{matrix} \right| = \int \left(t^{\frac{1}{3}} \cdot \frac{1}{2} \right) dt =$$

$$= \frac{1}{2} \cdot \int t^{\frac{1}{3}}\,dt = \frac{1}{2} \cdot \frac{t^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{1}{2} \cdot \frac{3}{4} t^{\frac{1}{3}} + C =$$

$$= \frac{3}{8}(2x-1)^{\frac{4}{3}} + C \quad ; \quad C \in R$$

ZAD: Izračunaj:

a) $$\int ctg\,x\,dx = \int \frac{\cos x}{\sin x}\,dx = \left| \begin{matrix} t = \sin x \\ dt = \cos x\,dx \end{matrix} \right| =$$

$$= \int \frac{dt}{t} = \ln|t| + C = \ln|\sin x| + C \quad ; \quad C \in R$$

b) $$\int tg\,x\,dx = \int \frac{\sin x\,dx}{\cos x} = \left| \begin{matrix} t = \cos x \\ dt = -\sin x\,dx \end{matrix} \right| = -1 \int \frac{dt}{t} =$$

$$= -\ln|t| + C = -\ln|\cos x| + C$$

ZAD: Izračunaj:

a) $$\int \frac{e^{2x}}{\sqrt{e^x+1}}\,dx = \left| \begin{matrix} t = e^x \\ dt = e^x\,dx \end{matrix} \right| = \int \frac{t\,dt}{\sqrt{t+1}} = \left| \begin{matrix} u = t+1 \\ t = u-1 \\ du = dt \end{matrix} \right| =$$

$$= \int \frac{(u-1)\,du}{\sqrt{u}} = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}} \right) du = \frac{u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C = \frac{2}{3}(t+1)$$

$$= \frac{2}{3}(t+1)^{\frac{3}{2}} - 2(t+1)^{\frac{1}{2}} + C = \frac{2}{3}(e^x+1)^{\frac{3}{2}} - 2(e^x+1)^{\frac{1}{2}} + C$$

$C \in R$

b) $$\int \frac{\ln^3 x}{x}\,dx = \left| \begin{matrix} t = \ln x \\ dt = \frac{1}{x}dx \end{matrix} \right| = \int t^3 \cdot dt =$$

$$= \frac{t^4}{4} + C = \frac{\ln^4 x}{4} + C$$

ZAD: Izračunaj:

$$\int \frac{\operatorname{arctg} x + x}{x^2+1}\,dx = \underbrace{\int \frac{\operatorname{arctg} x}{x^2+1}\,dx}_{I_1} + \underbrace{\int \frac{x}{x^2+1}\,dx}_{I_2}$$

$$I_1 = \left| \begin{matrix} t = \operatorname{arctg} x \\ dt = \frac{dx}{x^2+1} \end{matrix} \right| = \int t\,dt = \frac{t^2}{2} + C = \frac{\operatorname{arctg}^2 x}{2} + C$$

$C \in R$

$$I_2 = \left| \begin{matrix} t = x^2+1 \\ dt = 2x\,dx \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{t} = \frac{1}{2}\ln|t| + C = \frac{1}{2}\ln|x^2+1| + C$$

$$\int \frac{\operatorname{arctg} x + x}{x^2+1} = \frac{\ln|x^2+1| + \operatorname{arctg}^2 x}{2} + C \qquad C \in R$$

ZAD: Izračunaj:

$$\int \frac{x^9\,dx}{x^{10}-2x^5+5} = \left| \begin{matrix} x^5 = t \\ 5x^4\,dx = dt \end{matrix} \right| = \frac{1}{5}\int \frac{t\,dt}{t^2-2t+5} = \frac{1}{5}\int \frac{t\,dt}{(t-1)^2+4}$$

$t^2-2t+5 = t^2-2t\cdot 1+1+4 = (t-1)^2+4$

$$= \frac{1}{5}\int \frac{t\,dt}{(t-1)^2+4} = \left| \begin{matrix} u = t-1 \\ du = dt \end{matrix} \right| = \frac{1}{5}\int \frac{(u+1)\,du}{u^2+2} =$$

$$= \frac{1}{5}\underbrace{\int \frac{u\,du}{u^2+2^2}}_{I_1} + \frac{1}{5}\int \frac{du}{u^2+2^2} = \frac{1}{5}\cdot\frac{1}{2}\ln|u^2+4| + \frac{1}{5}\cdot\frac{1}{2}\operatorname{arctg}\left(\frac{u}{2}\right) + C =$$

$$I_1 = \left| \begin{matrix} w = u^2+2^2 \\ dw = 2u\,du \end{matrix} \right| = \frac{1}{2}\int \frac{dw}{w} = \frac{1}{2}\ln|w| + C = \frac{1}{2}\ln|u^2+4| + C$$

$$= \frac{1}{10}\ln\left|(t-1)^2+4\right| + \frac{1}{10}\operatorname{arctg}\left(\frac{t-1}{2}\right) + C$$

$$= \frac{1}{10}\ln\left|(x^5-1)^2+4\right| + \frac{1}{10}\operatorname{arctg}\left(\frac{x^5-1}{2}\right) + C$$

$C \in R$

PR: Izračunaj integral: (trigonometrijska supstitucija)

$$\int \sqrt{9-x^2}\,dx = \left| \begin{matrix} x = 3\cdot \sin t \\ dx = 3\cdot \cos dt \end{matrix} \right| = \int \sqrt{9-9\sin t}\cdot 3\cos t\,dt =$$

$$= 9\int \sqrt{1-\sin^2 t}\cdot \cos t\,dt = 9\int \sqrt{\cos^2 t}\cdot \cos t\,dt =$$

$$= 9\int |\cos t|\cdot \cos t\,dt =$$

↳ vratimo se na supstituciju:

$x = 3\sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$x \in [-3, 3]$ jer $9 - x^2 \geq 0$

$\sin t = \frac{x}{3}$

$\arcsin \frac{x}{3} = t$

$|\cos t| = \cos t$

$$= 9\int \cos^2 t\,dt = 9\int \frac{1+\cos 2t}{2}\,dt = 9\left[\int \frac{dt}{2} + \frac{1}{2}\underbrace{\int \cos 2t\,dt}_{\text{izvedi pokraj}}\right] =$$

$$= 9\left[\frac{1}{2}t + \frac{1}{2}\cdot\frac{1}{2}\sin(2t)\right] + C = \frac{9}{2}\arcsin\frac{x}{3} + \frac{3x}{2}\sqrt{1-\left(\frac{x}{3}\right)^2} + C$$

$C \in R$

ZAD: Izračunaj integral:

$$\int x^3 \operatorname{ch}(x^3+1)\,dx = \left| \begin{matrix} u = x^3+1 \\ du = 3x^2 dx \end{matrix} \right| = \frac{1}{3}\int \operatorname{ch} u\,du =$$

$$= \frac{1}{3}\operatorname{sh}(x^3+1) + C \quad ; \quad C \in R$$

ZAD: Izračunaj integral: (BITNO!)

$$\int \frac{dx}{\cos x} = ?$$

1. NAČIN

$$\int \frac{dx}{\cos x} = \int \frac{\cos x\, dx}{\cos^2 x} = \left| \begin{matrix} u = \sin x \\ du = \cos x\, dx \end{matrix} \right| = \int \frac{du}{1-u^2} =$$

$$= -\int \frac{du}{u^2-1} = (*)$$

↳ izvod tablični:

$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a} \ln\left|\frac{x-a}{x+a}\right| + C \; ; \quad C \in R \quad a>0$$

↳ rastav na parcijalne razlomke: (dobar tablični)

$$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$$

$$1 = A(u+1) + B(u-1)$$

$$1 = (A+B)u + (A-B) \quad \Rightarrow \quad \left.\begin{matrix} A+B=0 \\ A-B=1 \end{matrix}\right\} +$$

$$2A = 1$$

$$A = \frac{1}{2} \quad \Rightarrow B = -\frac{1}{2}$$

$$\frac{1}{u^2-1} = \frac{1}{2}\cdot\frac{1}{u-1} - \frac{1}{2}\cdot\frac{1}{u+1} //$$

↳ vratimo se na znak $(*)$:

$$(*) = -\left[\frac{1}{2}\int\frac{du}{u-1} - \frac{1}{2}\int\frac{du}{u+1}\right] = -\left[\frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1|\right] + C$$

$$= \frac{1}{2}\ln|u+1| - \frac{1}{2}\ln|u-1| + C$$

$$= \frac{1}{2}\ln\left|\frac{u+1}{u-1}\right| + C = \frac{1}{2}\ln\left|\frac{\sin x + 1}{\sin x - 1}\right| + C\,; \quad C \in R$$

2. NAČIN → univerzalna supstitucija !

$$\int\frac{dx}{\cos x} = ?$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\left(\frac{x}{2}\right)\cdot\left(1 - tg^2\left(\frac{x}{2}\right)\right)$$

$$\cos^2 x = \frac{1}{1+tg^2 x}$$

$$\cos x = \frac{1 - tg^2\frac{x}{2}}{1 + tg^2\frac{x}{2}}$$

$$\int\frac{dx}{\cos x} = \left|\begin{array}{l} t = tg\frac{x}{2} \\ dt = \frac{1}{\cos^2\frac{x}{2}}\cdot\frac{1}{2}dx \end{array}\right| = \int\frac{1+t^2}{1-t^2}\cdot 2\cdot\frac{1}{1+t^2}\cdot dt = (*)$$

→ $dx = 2\cos^2\frac{x}{2}\,dt = 2\cdot\frac{1}{1+t^2}\,dt$

$$(*) = 2\int\frac{dt}{1-t^2} = -2\int\frac{dt}{t^2-1} = -2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C =$$

$$tg(\alpha+\beta) = \frac{tg\alpha + tg\beta}{1 - tg\alpha\, tg\beta}$$

$$= \ln\left|\frac{t+1}{t-1}\right| + C = \ln\left|\frac{tg\frac{x}{2}+1}{tg\frac{x}{2}-1}\right| + C = \ln\left|tg\left(\frac{x}{2}+\frac{\pi}{4}\right)\right| + C =$$

= pomoću trigonometrijskih transformacija dobijemo isto kao u prvom načinu rješavanja

PARCIJALNE INTEGRACIJE

⇒ bitno jer ima velike posljedice u daljnjoj matematici:

$$\int f(x) \cdot g'(x)\,dx = \int (f(x) \cdot g(x))'\,dx - \int f'(x) \cdot g(x)\,dx$$

$$\boxed{\int f(x) \cdot g'(x)\,dx = f(x) \cdot g(x) - \int f'(x) \cdot g(x)\,dx}$$

↳ formula parcijalne integracije

↳ prebacuje deriviranje s jedne funkcije na drugu

⇒ STANDARDNE OZNAKE:

$u = f(x)$; $v = g(x)$

$du = f'(x)\,dx$; $dv = g'(x)\,dx$

$$\boxed{\int u\,dv = u \cdot v - \int v\,du}$$

↳ standardna formula parcijalne integracije

PR: $\int x \sin x \, dx = \left| \begin{matrix} u = x, & dv = \sin x \, dx \\ du = dx, & v = -\cos x \end{matrix} \right| = -x\cos x - \int (-\cos x)\, dx$

$= -x\cos x + \sin x + C, \quad C \in R$

PR: $\int \ln x \, dx = \left| \begin{matrix} u = \ln x, & dv = dx \\ du = \frac{dx}{x}, & v = x \end{matrix} \right| = x\ln x - \int x \cdot \frac{dx}{x} =$

$= x\ln x - x + C, \quad C \in R$

ZAD: $\int \arcsin x \, dx = \left| \begin{matrix} u = \arcsin x, & dv = dx \\ du = \frac{dx}{\sqrt{1-x^2}}, & v = x \end{matrix} \right| =$

$= x\arcsin x - \int x \frac{dx}{\sqrt{1-x^2}} = (*)$

$\int x \frac{dx}{\sqrt{1-x^2}} = \left| \begin{matrix} t = \sqrt{1-x^2} \\ dt = \frac{1}{2}(1-x^2)^{\frac{1}{2}} \cdot (-2x)\, dx \end{matrix} \right| = -\int dt =$

$= -t + C = -\sqrt{1+x^2} + C, \quad C \in R$

$(*) = x\arcsin x - \sqrt{1-x^2} + C, \quad C \in R$ //

ZAD: $\int e^x \sin x \, dx = \left| \begin{matrix} u = e^x, & dv = \sin x \, dx \\ du = e^x dx, & v = -\cos x \end{matrix} \right| =$

$= -e^x \cos x - \int (-\cos x) e^x \, dx =$

$= -e^x \cos x + \int e^x \cos x \, dx = \left| \begin{matrix} u = e^x, & dv = \cos x \, dx \\ du = e^x dx, & v = \sin x \end{matrix} \right| =$

$= -e^x \cos x + \left[e^x \cdot \sin x - \int \sin x \cdot e^x \, dx \right] =$

$= -e^x \cos x + e^x \sin x - \langle$ početni integral $\rangle$

↳ prebacimo početni integral na lijevu stranu:

$$2 \int e^x \sin x \, dx = e^x (\sin x - \cos x) + C \; ; \; C \in R$$

$$\int e^x \sin x \, dx = \frac{e^x}{2} (\sin x - \cos x)$$

ZAD: Odredi sve funkcije „f" čija derivacija je jednaka $x^2 \operatorname{arctg}\left(\frac{1}{x}\right)$ ∇

$$f(x) = \int x^2 \operatorname{arctg}\left(\frac{1}{x}\right) dx = \left| \begin{array}{l} u = \operatorname{arctg}\left(\frac{1}{x}\right), \; dv = x^2 dx \\ du = \frac{1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2} dx, \; v = \frac{x^3}{3} \end{array} \right| =$$

$$\frac{x^2}{1+x^2} = \frac{1}{\frac{1+x^2}{x^2}} = \frac{1}{1+\frac{1}{x^2}} \qquad \frac{x^2}{x^2+1} \cdot \frac{-1}{x^2}$$

$$= \frac{x^3}{3} \operatorname{arctg}\left(\frac{1}{x}\right) - \int \frac{x^3}{3} \cdot \frac{-1}{x^2+1} dx = \frac{x^3}{3} \operatorname{arctg}\left(\frac{1}{x}\right) + \frac{1}{3} \underbrace{\int \frac{x^3}{x^2+1} dx}_{I}$$

$$I = \int \frac{x^3 dx}{x^2+1} = \int \frac{(x^2+1)x - x}{(x^2+1)} dx = \int x\,dx - \int \frac{x\,dx}{x^2+1} = \left| \begin{array}{l} t = x^2+1 \\ dt = 2x\,dx \end{array} \right| =$$

$$\begin{array}{r} x^3 : (x^2+1) = x \\ -x^3 \pm x \\ \hline -x \end{array} \Bigg\} \; x^3 = (x^2+1)x - x$$

$$I = \frac{x^2}{2} - \frac{1}{2} \int \frac{dt}{t} = \frac{x^2}{2} - \frac{1}{2} \ln|x^2+1| + C, \; C \in R$$

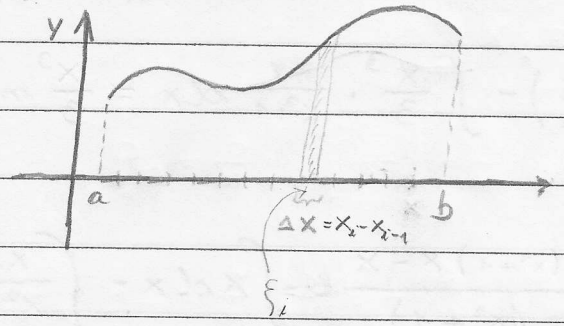
$$f(x) = \frac{x^3}{3} \operatorname{arctg}\left(\frac{1}{x}\right) + \frac{1}{3} \cdot \frac{x^2}{3} - \frac{1}{6} \ln(x^2+1) + C$$

$C \in R$

ODREĐENI INTEGRALI

⇒ koristo za:

(i) npr. računanje rada $W = F \cdot d$, ako sila nije konstanta



$$W_i \approx f(\xi_i)\,\Delta x$$

↑ aproksimacija

$$\xi_i \in \langle x_{i-1}, x_i \rangle$$

↳ ukupni rad:

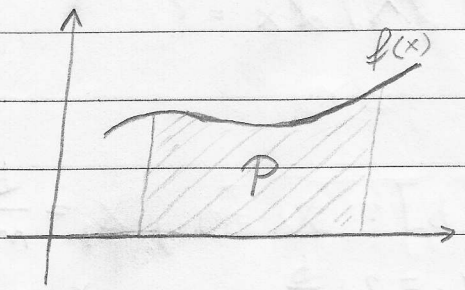
$$\sum_{i=1}^{n} W_i \rightsquigarrow \underbrace{\sum_{i=1}^{n} f(\xi_i)\,\Delta x}_{\text{integralna suma}}$$

↳ veći „n" znači bolja aproksimacija ∇

↳ zaključujemo:

$$W = \lim_{n \to \infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta x = \int_a^b f(x)\,dx$$

(ii) npr. računanje površine lika ispod grafa funkcije ako je ($f(x) \geq 0$)



1.) rastavi na pravokutnike

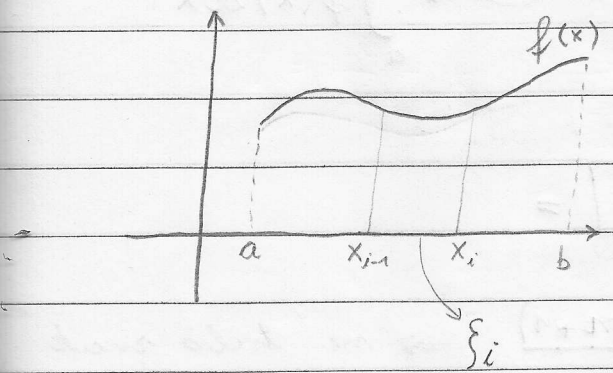
2.) zbroj površine pravokutnika

3.) uzmi limes kada pravokutnici postaju sve tanji

⇒ NEWTON - LEIBNITZOVA FORMULA:

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad F' = f$$

(ii) → nastavak:

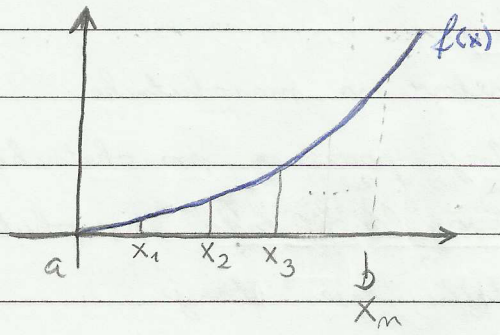


$$m_i \leq f(\xi_i) \leq M_i$$

m_i = min „ f “ na $[x_{i-1}, x_i]$

M_i = max „ f “ na $[x_{i-1}, x_i]$

PR: Odredi integral: $f(x) = x^2$, $a = 0$, $b > 0$ (površina)



$$\int_a^b x^2\,dx = ?$$

· $[0, b]$: $x_0 = 0$, $x_1 = \frac{b}{n}$,
$x_2 = 2 \cdot \frac{b}{n}$, ..., $x_n = n \cdot \frac{b}{n} = b$

$$P_{\square} = \frac{b}{n} \cdot f\left(\frac{b}{n}\right) + \frac{b}{n} \cdot f\left(\frac{2b}{n}\right) + \ldots + \frac{b}{n} \cdot f(b)$$

$$P_{\square} = \frac{b}{n} \cdot \left(\frac{b}{n}\right)^2 + \frac{b}{n} \cdot \left(\frac{2b}{n}\right)^2 + \ldots + \frac{b}{n} \cdot \left(\frac{nb}{n}\right)^2 = (*)$$

↳ $P_{\square}$ je zapravo integralna suma od „f" na intervalu $[a, b]$:

$$\sum_{i=1}^{n} f(\xi_i)\,\Delta x \xrightarrow{\lim_{n\to\infty}} \int_a^b f(x)\,dx$$

$$(*) = \left(\frac{b}{n}\right)^3 \cdot \underbrace{\left[1 + 2^2 + 3^2 + \ldots + n^2\right]}_{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}} =$$

→ ne treba znati napamet (pisat će)

$$= \left(\frac{b}{n}\right)^3 \cdot \frac{n(n+1)(2n+1)}{6}$$

↳ uvrstimo sada limes:

$$\lim_{n\to\infty} \left(\frac{b}{n}\right)^3 \cdot \frac{n(n+1)(2n+1)}{6} = \frac{b^3}{3} = \int_a^b x\,dx$$

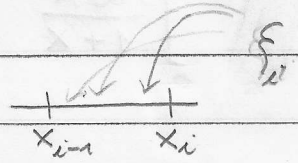
↳ za lakše računanje integrala od ovog u prošlom primjeru (po definiciji), koristi se Newton - Leibnitzova formula:

$$\int_a^b f(x)\,dx = F(b) - F(a) \quad ; \quad F' = f$$

ZAD: Izračunaj $\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+\frac{2i}{n}} \cdot \frac{2}{n}$ ▽. (TEST!)

↳ treba prepoznat integralnu sumu

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\,\Delta x$$



$$\Delta x = \frac{2}{n} = \frac{b-a}{n} \qquad b = 2 \quad a = 0$$

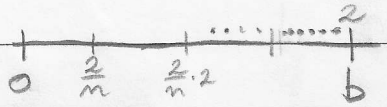
→ to je naš ξ_i

$$f(\xi_i) = \sqrt{1+\frac{2}{n}\cdot i}$$

granica

pretpostavka

↳ znači, mi smo podijelili segment:



↳ eto, sad smo nanjušil granice našeg integrala: $b = 2$, $a = 0$

$$\int_0^2 \sqrt{1+x}\,dx = \left| \begin{matrix} t = 1+x \\ dt = dx \\ x=0 \;\; t=1 \\ x=2 \;\; t=3 \end{matrix} \right| = \int_1^3 t^{\frac{1}{2}}\,dt = \frac{t^{\frac{3}{2}}}{\frac{3}{2}}\Bigg|_1^3 =$$

$$= \frac{2}{3}\left(\sqrt{27} - \sqrt{1}\right) = \frac{2}{3}\left(3\sqrt{3} - 1\right)$$

ZAD: Rješi sljedeće zadatke:

a) $\lim_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\dots+\frac{1}{n+n}\right]$

↳ treba prepoznat $\int$: (najvažnije)

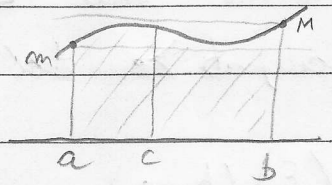
$$\frac{1}{n\cdot\left(1+\frac{1}{n}\right)}+\frac{1}{n\cdot\left(1+\frac{2}{n}\right)}+\dots+\frac{1}{n\left(1+\frac{n}{n}\right)}=$$

$$=\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\dots+\frac{1}{1+\frac{n}{n}}\right]$$



= bitan teorem:

↳ neka je „f" neprekidna na $[a, b]$



$f > 0 \qquad \frac{1}{b-a} \int_a^b f(x)\,dx$

↳ ako je „f" neprekidna na $[a, b]$, onda je ograničena na $[a, b]$:

$$m \le f(x) \le M \,; \quad \forall x \in [a, b]$$

$$m \int_a^b dx \le \int_a^b f(x)\,dx \le M \int_a^b dx$$

$$m \cdot (b-a) \le \int_a^b f(x)\,dx \le M \cdot (b-a) \quad / : (b-a)$$

$$\boxed{m \le \frac{1}{(b-a)} \int_a^b f(x)\,dx \le M}$$

↳ kako je „f" neprekinuta na $[a, b]$, onda ona poprima minimum i maksimum vrijednosti na $[a, b]$, kao i svaku međuvrijednost (to je svojstvo neprekidne funkcije na segmentu)

↳ ZAKLJUČAK:

$$\exists \xi \in [a, b] \text{ td } f(\xi) = \frac{1}{b-a} \int_a^b f(x)\,dx$$

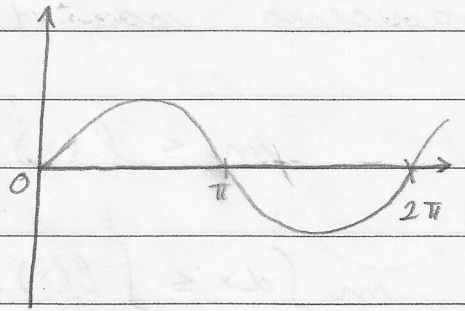
⇒ TEOREM SREDNJE VRIJEDNOSTI INTEGRALNOG RAČUNA:

↳ neka je „f" neprekidna na $[a, b]$

↳ tada postoj $\xi \in \langle a, b \rangle$ td:

$$\int_a^b f(x)\,dx = f(\xi)(b-a)$$

PR: $\int_0^{2\pi} \sin x\,dx = ?$



$$\int_0^{2\pi} \sin x\,dx = -\cos x \Big|_0^{2\pi} =$$

$$= -(\cos 2\pi - \cos 0) = 0$$

$$\frac{1}{2\pi} \int_0^{2\pi} \sin x\,dx = 0 = f(\xi)$$

↳ ξ može biti: $\xi = 0$, $\xi = \pi$, $\xi = 2\pi$

NEWTON - LEIBNITZOVA FORMULA

⇒ dokaz se ovaj put na testu

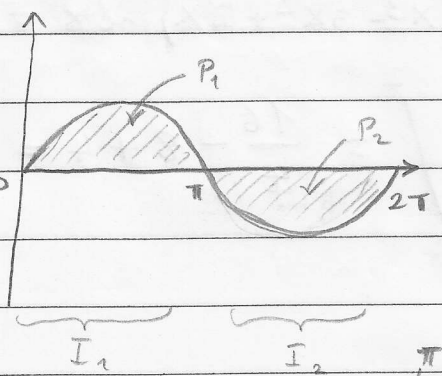
- „f" je neprekinuta na $[a, b]$
- „F" je bilo koja primitivna funkcija od „f" na intervalu $\langle a, b \rangle$

↳ tada je:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

PR:

$$\int_0^{2\pi} \sin x\,dx = 0$$



$$P_{UK} = P_1 + P_2$$

$$I_1 = \int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = -(\cos\pi - \cos 0) = 2 \;;\quad P_1 = I_1 = 2$$

$$I_2 = \int_{\pi}^{2\pi} \sin x\,dx = -\cos x \Big|_{\pi}^{2\pi} = -(\cos 2\pi - \cos\pi) = -2 \;;\quad P_2 = -I_2 = 2$$

$$P_{UK} = 4$$

$$I = I_1 + I_2 = 0$$

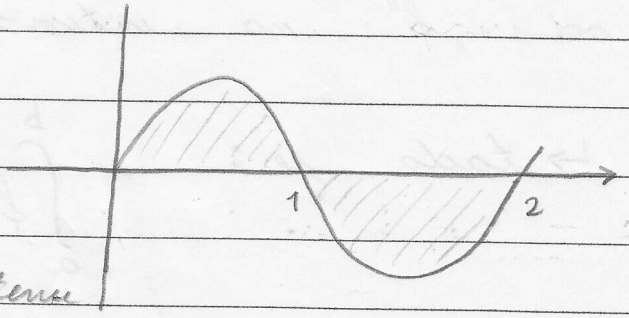
} PAZI !

⇒ geometrijska interpretacija odredenog integrala:

- pozitivna površina kada je funkcija iznad x-osi
- negativna površina kada je funkcija ispod x-osi

ZAD: $f(x) = x(x-1)(x-2)$

$$I = \int_0^2 f(x)\,dx$$



↳ odred površinu omeđenu $G(f)$ i x-osi

$$I = \int_0^2 x(x^2-3x+2)\,dx = \int_0^2 (x^3-3x^2+2x)\,dx =$$

$$= \left[\frac{x^4}{4} - 3\cdot\frac{x^3}{3} + 2\cdot\frac{x^2}{2}\right]\Bigg|_0^2 = \frac{16}{4} - 8 + 4 = 0$$

$$P = P_1 + P_2$$

$$P_1 = \int_0^1 x(x-1)(x-2)\,dx = \ldots = \frac{1}{4}$$

$$P_2 = -I_2 = -\int_1^2 x(x-1)(x-2)\,dx = \ldots = -\left(-\frac{1}{4}\right) = \frac{1}{4}$$

$$P = P_1 + P_2 = \frac{1}{2}$$

⇒ još jedan teorem koji će pomoći u dokazu Newton - Leibnitzove formule:

- „f" je neprekidna
- $G(x) = \int_a^x f(t)\,dt$, $a \le t \le x$

↳ tada je: $G'(x) = f(x)$

PR: $\frac{d}{dx}\left[\int_1^x \frac{dt}{t^2}\right] = \frac{1}{x^2}$

↳ $G(x)$

$f(t) = \frac{1}{t^2}$

→ to možemo dobit i pješce računanjem integrala, pa deriviranjem

$\frac{d}{dx}\left[\int_1^{x^4} \ln t\,dt\right]$

INTEGRALI RACIONALNE FUNKCIJE

$$\int \frac{P_n(x)}{Q_m(x)}\,dx$$

$P_n \rightarrow$ polinom n-tog stupnja

$Q_m \rightarrow$ polinom m-tog stupnja

BITNO!

↳ za to integriranje, stupanj nazivnika treba biti veći od stupnja brojnika ($m > n$)

↳ ako nije, podijeli polinome!

↳ faktoriziramo $Q_m(x)$ te radimo rastav na parcijalne razlomke!

PR: $$\frac{-2x^3+2x-7}{(x^2-x+1)^3\cdot(x^2+1)(x-3)^2(x-1)} = \frac{Ax+B}{x^2-x+1} + \frac{Cx+D}{(x^2-x+1)^2} + \frac{Ex+F}{(x^2-x+1)^3} +$$

$$+ \frac{Gx+H}{x^2+1} + \frac{I}{x-3} + \frac{J}{(x-3)^2} + \frac{K}{x+1}$$

↳ kad imamo realne nultočke, gore je samo konstanta, a kad imamo kompleksne nultočke, gore je polinom 1. stupnja

↳ sada vidimo, imamo sumu lakših integrala umjesto velike racionalne funkcije

Primjer 3!

↳ uzmimo ovaj:

$$\int \frac{Ex+F}{(x^2-x+1)^3}\,dx = \int \frac{Ex+F}{\left[(x-\frac{1}{2})^2+\frac{3}{4}\right]^3}\,dx = E\underbrace{\int \frac{x+\frac{F}{E}}{\left[(x-\frac{1}{2})^2+\frac{3}{4}\right]^3}\,dx}_{I} = (*)$$

nije bitan pa kako ne pišemo $\frac{F}{E}$, stavimo λ

$$I = \left|\begin{matrix} t = x-\frac{1}{2} \\ dt = dx \end{matrix}\right| = \int \frac{t+\frac{1}{2}+\lambda}{(t^2+\frac{3}{4})^3}\,dt = \underbrace{\int \frac{t\,dt}{(t^2+\frac{3}{4})^3}}_{I^{(*)}} + (\lambda+\frac{1}{2})\underbrace{\int \frac{dt}{(t^2+\frac{3}{4})^3}}_{I^{(**)}}$$

$$I^{(*)} = \left|\begin{matrix} u = t^2+\frac{3}{4} \\ du = 2t\,dt \end{matrix}\right| = \frac{1}{2}\int \frac{du}{u^2} = \ldots\ldots$$

$I^{(**)}$ = oblik $\int \frac{dt}{(t^2+a^2)^m}\,dt$ što nećemo raditi jer je to rekurzivni oblik koji za nas nije bitan kako se rješava

↳ taka Wolfram računa ▽

⇒ integral koji se dopunjavaju na potpuni kvadrat uvijek razvijaju „$\arctan\left(\frac{x}{a}\right)$" jer u nazivniku uvijek imam $x^2 + a^2$ (tj. uvijek imam zbroj kvadrata) BITNO!▽

Zad: $\int \frac{dx}{x\ln^3 x + 4x\ln x} = \left| \begin{matrix} u = \ln x \\ du = \frac{dx}{x} \end{matrix} \right| = \int \frac{du}{u^3+4u} = \int \frac{du}{u(u^2+4)} = (*)$

$$\frac{1}{u(u^2+4)} = \frac{A}{u} + \frac{Bu+C}{u^2+4}$$

$$1 = A(u^2+4) + Bu^2 + Cu$$

$$1 = Au^2 + 4A + Bu^2 + Cu$$

$$1 = u^2(A+B) + Cu + 4A$$

$$\hookrightarrow A = \frac{1}{4},\ B = -\frac{1}{4},\ C = 0$$

$$(*) = \frac{1}{4}\int \frac{du}{u} + \frac{1}{4}\int \frac{u\,du}{u^2+4} = \frac{1}{4}\ln|u| - \frac{1}{4}\int \frac{u\,du}{u^2+4} =$$

$$= \left| \begin{matrix} t = u^2+4 \\ dt = 2u\,du \end{matrix} \right| = \frac{1}{4}\left(\ln|u| - \frac{1}{2}\int \frac{dt}{t}\right) = \frac{1}{4}\left(\ln|u| - \frac{1}{2}\ln|t|\right) + C =$$

$$= \frac{1}{4}\left(\ln|u| - \frac{1}{2}\ln|u^2+4|\right) + C = \frac{1}{4}\left(\ln|\ln x| - \frac{1}{2}\ln|\ln^2 x + 4|\right) + C$$

$C \in \mathbb{R}$

INTEGRIRANJE NEKIH IRACIONALNIH FUNKCIJA

ZAD: $\int_1^3 \sqrt{3+2x-x^2}\,dx = (*)$

↳ nadopuni na puni kvadrat:

$$3+2x-x^2 = -(x^2-2x-3) = -(x^2-2x\cdot 1+1-4) =$$
$$= -((x-1)^2-4) = 4-(x-1)^2$$

$$(*) = \int_1^3 \sqrt{4-(x-1)^2}\,dx = \left| \begin{matrix} t = x-1 \\ x=1,\ t=0 \\ x=3,\ t=2 \end{matrix} \right| = \int_0^2 \sqrt{4-t^2}\,dt =$$

ZAPAMTI ▽ ZA SINUS°

$$= \left| \begin{matrix} t = 2\sin u \\ dt = 2\cos u\,du \\ u(0)=0 \\ u(2)=\frac{\pi}{2} \end{matrix} \right| = \int_0^{\frac{\pi}{2}} \sqrt{4\cos^2 u}\cdot 2\cos u\,du = 4\int_0^{\frac{\pi}{2}} |\cos u|\cos u\,du =$$

$$= 4\int_0^{\frac{\pi}{2}} \cos^2 u\,du = 4\int_0^{\frac{\pi}{2}} \frac{1+\cos 2u}{2}\,du = 2\left[\int_0^{\frac{\pi}{2}} du + \int_0^{\frac{\pi}{2}} \cos 2u\,du\right]$$

$$= \ldots\ldots = \pi$$

ZAD: $\int \frac{x\,dx}{\sqrt{2+x-x^2}} = (*)$

↳ nadopisuję do pełnego kwadratu:

$$2+x-x^2 = -(x^2-x-2) = -\left[\left(x-\frac{1}{2}\right)^2 - \frac{9}{4}\right]$$

$$(*) = \int \frac{x\,dx}{\sqrt{\frac{9}{4}-\left(x-\frac{1}{2}\right)^2}} = \left|\begin{matrix} t = x-\frac{1}{2} \\ dt = dx \end{matrix}\right| = \int \frac{t+\frac{1}{2}}{\sqrt{\frac{9}{4}-t^2}}\,dx = \underbrace{\int \frac{t\,dt}{\sqrt{\frac{9}{4}-t^2}}}_{I_1} + \frac{1}{2}\underbrace{\int \frac{dt}{\sqrt{\frac{9}{4}-t^2}}}_{I_2}$$

$$I_1 = \left|\begin{matrix} u = \frac{9}{4}-t^2 \\ du = -2t\,dt \end{matrix}\right| = -\frac{1}{2}\int \frac{du}{\sqrt{u}} = -\frac{1}{2}\cdot\frac{\sqrt{u}}{\frac{1}{2}} + C = -\sqrt{u} + C =$$

$$= -\sqrt{\frac{9}{4}-t^2} + C = -\sqrt{\frac{9}{4}-\left(x-\frac{1}{2}\right)^2} = -\sqrt{2+x-x^2} + C$$

$$I_2 = \int \frac{dt}{\sqrt{\frac{9}{4}-t^2}} = \arcsin\left(\frac{2t}{3}\right) + C = \arcsin\left(\frac{2x-1}{3}\right) + C$$

↳ rezultat:

$$I_1 + \frac{1}{2} I_2$$

ZAD: Nađi $b \in \mathbb{R}$, tako da vrijedi:

$$\int_{\ln 2}^{b} \frac{dx}{\sqrt{e^x - 1}} = \frac{\pi}{6}$$

PROUČI ☝

↳ integriramo:

$$\int_{\ln 2}^{b} \frac{dx}{\sqrt{e^x-1}} = \left| \begin{array}{l} t = \sqrt{e^x - 1} \\ t^2 = e^x - 1 \end{array} \right. \rightarrow e^x = t^2 + 1 \rightarrow x = \ln(t^2+1) \rightarrow dx = \frac{2t\,dt}{t^2+1}$$

$$x = \ln 2, \quad t = \sqrt{2-1} = 1$$

$$x = b, \quad t = \sqrt{e^b - 1}$$

$$= \int_{1}^{\sqrt{e^b-1}} \frac{1}{t} \cdot \frac{2t\,dt}{t^2+1} = 2 \int_{1}^{\sqrt{e^b-1}} \frac{dt}{t^2+1} = 2 \operatorname{arctan} t \Big|_{1}^{\sqrt{e^b-1}} =$$

$$= 2\left(\operatorname{arctan}\sqrt{e^b-1} - \frac{\pi}{4}\right)$$

$$2 \operatorname{arctan}\sqrt{e^b-1} - \frac{\pi}{2} = \frac{\pi}{6}$$

$$\operatorname{arctan}\sqrt{e^b-1} = \frac{\pi}{3}$$

$$\sqrt{e^b-1} = \sqrt{3}$$

$$e^b - 1 = 3$$

$$e^b = 4$$

$$b = \ln 4$$

$$b = 2 \ln 2$$

ZAD: $\int_1^{\sqrt{2}} \frac{x\,dx}{\sqrt{3-x^2}} = \left| \begin{matrix} t = 3 - x^2 \\ dt = -2x\,dx \\ t(1) = 2,\ t(\sqrt{2}) = 1 \end{matrix} \right| = -\frac{1}{2} \int_2^1 \frac{dt}{\sqrt{t}} =$

$= \frac{1}{2} \int_1^2 t^{-\frac{1}{2}}\,dt = \frac{1}{2} \cdot \frac{t^{\frac{1}{2}}}{\frac{1}{2}} \Big|_1^2 = \sqrt{t} \Big|_1^2 = \sqrt{2} - 1$

↳ 2 način:

$\int_1^{\sqrt{2}} \frac{x\,dx}{\sqrt{3-x^2}} = \left| \begin{matrix} t = \sqrt{3-x^2} \\ dt = \frac{-2x\,dx}{2\sqrt{3-x^2}} \end{matrix} \quad \cdots\cdots\cdots \right| =$

$= \int_{\sqrt{2}}^1 (-dt) = \int_1^{\sqrt{2}} dt = \sqrt{2} - 1$

ZAD: $\int_{-1}^{1} x \sqrt{\frac{4-x^2}{4+x^2}}\,dx = 0$

↳ neparna funkcija na simetričnom intervalu oko nule pa je rješenje nula

ZAD 3: $\int \sqrt{\frac{x}{3-x}}\,dx = \left| \begin{matrix} t^2 = \frac{x}{3-x} \\ t^2(3-x) = x \end{matrix} \rightarrow \begin{matrix} x = \frac{3t^2}{1+t^2} \\ dx = \frac{6t\,dt}{(1+t^2)^2} \end{matrix} \right| =$

$$= \int t \cdot \frac{6t}{(1+t^2)^2}\,dt = 6 \underbrace{\int \frac{t^2}{(1+t^2)^2}\,dt}_{I} = (*)$$

$$I = \int \frac{t^2\,dt}{(t^2+1)^2} = \left| \begin{matrix} u = t \\ du = dt \end{matrix} \quad \begin{matrix} v = \int \frac{t\,dt}{(t^2+1)^2} = \left| \begin{matrix} w = t^2+1 \\ dw = 2t\,dt \end{matrix} \right| = \ldots = -\frac{1}{2(t^2+1)} \\ dv = \frac{t\,dt}{(t^2+1)^2} \end{matrix} \right| =$$

$$= -t \cdot \frac{1}{2(t^2+1)} - \int \left(-\frac{1}{2(t^2+1)}\right) dt = -\frac{t}{2(t^2+1)} + \frac{1}{2} \int \frac{dt}{t^2+1} =$$

$$= -\frac{t}{2(t^2+1)} + \frac{1}{2} \operatorname{arctg} t + C$$

↓

vrat supstituciju „t" i vrat se u glavni integral !?

ZAD: $\int_0^1 \frac{1+\sqrt{x}}{1+\sqrt[3]{x}}\,dx = \left|\begin{matrix} x=t^6 \\ dx=6t^5dt \\ t(0)=0;\ t(1)=1 \end{matrix}\right| = 6\int_0^1 \frac{1+t^3}{1+t^2}\,t^5\,dt = 6\int_0^1 \frac{t^8+t^5}{t^2+1}\,dt =$

$= 6\int_0^1 (t^6-t^4+t^3+t^2-t-1)\,dt + 6\int_0^1 \frac{t+1}{t^2+1}\,dt$

↳ sada je dalje lagano

⇒ NAPOMENA: „mali trik"

↳ PR: $\int \frac{x+1}{x^2+x+1}\,dx = (*)$

$u = x^2+x+1$

$du = (2x+1)\,dx$ → u brojniku želimo $2x$, pa

možemo to nastimati:

$(*) = \frac{1}{2}\int \frac{2x+2}{x^2+x+1}\,dx = \frac{1}{2}\int \frac{2x+1+1}{x^2+x+1}\,dx =$

$= \frac{1}{2}\int \frac{2x+1}{x^2+x+1}\,dx + \frac{1}{2}\int \frac{1}{x^2+x+1}\,dx$

lagan (na puni kvadrat)

ovdje supstitucija:

$u = x^2+x+1$

INTEGRALI NEKIH TRIGONOMETRIJSKIH I HIPERBOLIČKIH FUNKCIJA

⇒ univerzalna supstitucija:

$$t = \operatorname{tg}\frac{x}{2} \Rightarrow x = 2\operatorname{arctg} t$$

$$dx = \frac{2\,dt}{1+t^2}$$

a) kako sinus izražavamo pomoću tangensa?

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\cos^2\left(\frac{x}{2}\right)\cdot\operatorname{tg}\frac{x}{2} = \frac{2t}{1+t^2}$$

b) kako kosinus izražavamo pomoću tangensa?

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \cos^2\frac{x}{2}\left(1-\operatorname{tg}^2\frac{x}{2}\right) = \frac{1-t^2}{1+t^2}$$

⇒ NAPOMENA: integral oblika $\int \sqrt{2+x}\,dx$ rješava se tako da supstituiramo hiperboličkom funkcijom „sh t"; $x = \sqrt{2}\,\operatorname{sh} t$

Zad. $\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x \cos^3 x\,dx = \int_{\frac{1}{2}}^{\frac{\pi}{2}} \sin^2 x \cos^2 x \cos x\,dx =$

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^2 x (1-\sin^2 x)\cos x\,dx = \left| \begin{array}{l} t = \sin x \\ dt = \cos x\,dx \\ t\left(\frac{\pi}{6}\right) = \frac{1}{2};\ t\left(\frac{\pi}{2}\right) = 1 \end{array} \right| = \int_{\frac{1}{2}}^{1} t^2(1-t^2)\,dt =$$

$$= \frac{47}{480}$$

ZAD: $\int \frac{dx}{\sin^2 x + \sin x \cdot \cos x + 2\cos^2 x} = (*)$

↳ možemo i univerzalnom supstitucijom, ali je komplikovano pa pokušamo ovako:

$$\sin^2 x + \sin x \cos x + 2\cos^2 x = \cos^2 x \left(\operatorname{tg}^2 x + \operatorname{tg} x + 2\right) =$$

$$= \frac{\operatorname{tg}^2 x + \operatorname{tg} x + 2}{1 + \operatorname{tg}^2 x}$$

$$(*) = \left| \begin{array}{l} t = \operatorname{tg} x \quad x = \operatorname{arctg} t \\ dx = \frac{dt}{1+t^2} \end{array} \right| = \int \frac{\frac{dt}{1+t^2}}{\frac{t^2+t+2}{1+t^2}} = \int \frac{dt}{t^2+t+2} =$$

= ..... = (nadopuni do punog kvadrata) $= \frac{2}{\sqrt{7}} \operatorname{arctg}\left(\frac{2\operatorname{tg} x + 1}{\sqrt{7}}\right) + C$

ZA DO: $\int \frac{dx}{\sin x \sin 2x} = \int \frac{dx}{\sin x \cdot 2\sin x \cos x} = \frac{1}{2}\int \frac{dx}{\sin^2 x \cos x} = \frac{1}{2}\int \frac{\cos x\, dx}{\sin^2 x \cos^2 x} =$

$= \left| \begin{matrix} t = \sin x \\ dt = \cos x\, dx \end{matrix} \right| = \frac{1}{2}\int \frac{dt}{t^2(1-t^2)} = -\frac{1}{2}\int \frac{dt}{t^2(t^2-1)} =$

↳ razstav na parcialne ulomke

⇒ BITNI TRIKOVI ZA NEKE OBLIKE:

a) $f(x) = \sin x \cos 3x \sin 3x$

$= \frac{1}{2} \sin x \sin 6x$

$\sin\alpha \cdot \sin\beta = \ldots$ formula

b) $f(x) = \sin^3 x \cos^4 x = \sin x \sin^2 x \cos^4 x = \left| \begin{matrix} u = \cos x \\ du = -\sin x\, dx \end{matrix} \right|$

c) $f(x) = \sin^4 x ; \cos^4 x$

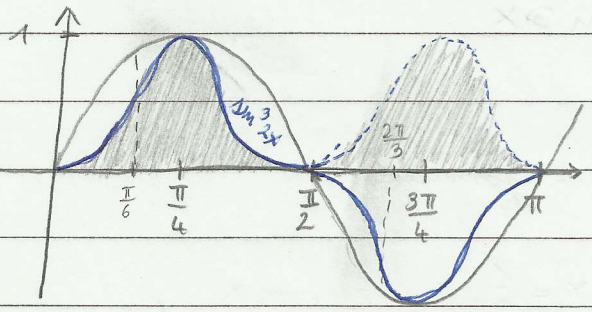
$(\sin^2 x)^2 = \left(\frac{1-\cos 2x}{2}\right)^2 = \frac{1}{4}\left(1 - 2\cos 2x + \cos^2 2x\right)$

ZADO: $\int_0^{100\pi} \sqrt{1-\cos(2x)}\,dx = \int_0^{100\pi} \sqrt{2\sin^2 x}\,dx = \sqrt{2}\int_0^{100\pi} |\sin x|\,dx =$

$= \sqrt{2} \cdot 100 \int_0^{\pi} \sin x\,dx = \sqrt{2}\cdot 100\,(-\cos x)\Big|_0^{\pi} =$

$= 200\sqrt{2}$

ZADO: $\int_{\frac{\pi}{6}}^{\frac{2\pi}{3}} |\sin(2x)|^3\,dx = \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^3(2x)\,dx + \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} (-\sin^3(2x))\,dx =$



$= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \sin^3(2x)\,dx - \int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} \sin^3(2x)\,dx =$

$\int \sin^3(2x)\,dx = \int (1-\cos^2(2x))\sin x\,dx =$

$= \left| \begin{matrix} t=\cos 2x \\ dt=-2\sin 2x \end{matrix} \right| = \frac{1}{2}\int (t^2-1)\,dt = \frac{1}{2}\left(\frac{t^3}{3}-t\right) + C =$

$= \frac{1}{2}\left(\frac{\cos^3 2x}{3} - \cos(2x)\right) + C$

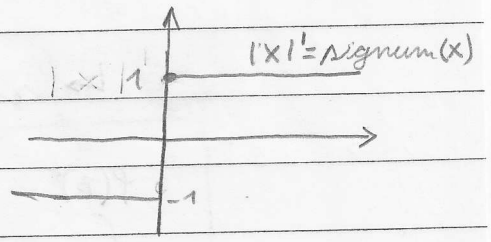
↳ sada se s tim integralom vratimo gore

⇒ NAPOMENA:

$$\int \frac{dx}{\sqrt{x^2+A}} = \ln\left|x+\sqrt{x^2+A}\right| + C, \quad A \neq 0$$

↳ dokaz:

$$\left|\ln\left|x+\sqrt{x^2+A}\right|\right|' = ?$$



$$|x|' = \frac{x}{|x|} = \operatorname{sgn}(x)$$

$$|f(x)|' = (\operatorname{sgn}(f(x)) \cdot f'(x)$$

$$f(x) = x+\sqrt{x^2+A}$$

$$\left(\ln|f(x)|\right)' = \frac{1}{|f(x)|}\cdot|f(x)|' = \frac{1}{|f(x)|}\cdot\operatorname{sgn}(f(x))\cdot f'(x) =$$

$$= \frac{1}{|f(x)|}\cdot\frac{f(x)}{|f(x)|}\cdot f'(x) = \frac{f(x)}{f^2(x)}\cdot f'(x) =$$

$$= \frac{f'(x)}{f(x)} = (*)$$

$$f'(x) = 1+\frac{2x}{2\sqrt{x^2+A}} = \frac{x+\sqrt{x^2+A}}{\sqrt{x^2+A}}$$

$$(*) = \frac{\frac{x+\sqrt{x^2+A}}{\sqrt{x^2+A}}}{x+\sqrt{x^2+A}} = \frac{1}{\sqrt{x^2+A}}$$

↳ DOKAZANO.

⇒ **NAPOMENA:**

npr. $\int_0^{\frac{1}{2}} \sqrt{1-x^2}\,dx = \left| \begin{array}{l} x = \sin t = \varphi(t) \\ dx = \varphi'(t)dt = \cos t\,dt \\ x=0 \to 0=\varphi(\alpha),\ \alpha=0 \\ x=\frac{1}{2},\ \frac{1}{2}=\varphi(\beta),\ \beta=\frac{\pi}{6} \end{array} \right| = \int_0^{\frac{\pi}{6}} \sqrt{1-\sin^2 t}\,\cos t\,dt =$

↳ ako poopćimo:

$$\int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx = \int_{\alpha}^{\beta} f\big(\varphi(t)\big)\cdot\varphi'(t)\,dt$$

↳ ova je općenita formula metode supstitucije koju smo već puno puta koristili

↳ dokaz:

- „F" je primitivna funkcija od „f" na I
- $f\circ\varphi$ dobro definirana: φ definirana na J, $\varphi(J)\subseteq I$

⇒ • $G = F\circ\varphi$ primitivna funkcija od $(f\circ\varphi)\varphi'$ na J

⇒ zaista: $G'(x) = F'\big(\varphi(x)\big)\,\varphi'(x) = f\big(\varphi(x)\big)\cdot\varphi'(x)$

slijed

$$\int_{\alpha}^{\beta} f\big(\varphi(t)\big)\cdot\varphi'(t)\,dt = \int_{\alpha}^{\beta} G'(x)\,dx = G(\beta)-G(\alpha) =$$

$$= F\big(\varphi(\beta)\big) - F\big(\varphi(\alpha)\big) = \int_{\varphi(\alpha)}^{\varphi(\beta)} f(x)\,dx$$

↳ DOKAZANO!

NEPRAVI INTEGRALI

⇒ integral s beskonačnim područjem integracije il na području integracije funkcija ima prekid

npr. • $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$ ⇒ primjena u statistici

• $\Gamma(t) = \int_{0}^{+\infty} x^{t-1} \cdot e^{-x} dx$ ⇒ proširenje faktorijela

• $\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ik \cdot x} dx$ ⇒ Fourijera transformacija

• $F(P) = \int_{0}^{\infty} e^{-Pt} f(t)\, dt$ ⇒ Laplaceova transformacija

⇒ NEPRAVI INTEGRALI PRVE VRSTE:

↳ integral neprekidne funkcije na beskonačnom intervalu

⇒ NEPRAVI INTEGRAL DRUGE VRSTE:

↳ integral funkcije na konačnom intervalu na kojem funkcija ima prekid

NEPRAVI INTEGRALI PRVE VRSTE

$$\int_a^{\infty} f(x)\,dx$$

↳ $a \in \mathbb{R}$, f je integrabilna na svakom segmentu $[a, L]$ $L < \infty$

↳ pustimo L na limes:

$$\lim_{L\to\infty} \int_a^{L} f(x)\,dx \quad \left(= \int_a^{\infty} f(x)\,dx \right)$$

↳ analogno:

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{l\to-\infty} \int_l^{b} f(x)\,dx$$

⇒ zaključak:

$$\boxed{\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\substack{l\to-\infty \\ L\to\infty}} \int_l^{L} f(x)\,dx}$$

↳ „l", „L" idu u svoje beskonačnosti različitim brzinama

PR: $\int_1^\infty \frac{dx}{x} = \lim_{L\to\infty} \int_1^L \frac{dx}{x} = \lim_{L\to\infty} \ln|x| \Big|_1^L = \lim_{L\to\infty} (\ln L - \ln 1) = +\infty$

↳ ovaj integral divergira

PR: $\int_0^\infty e^{-kx}\,dx = \lim_{L\to\infty} \int_1^L e^{-kx}\,dx = (*)$

$$\int e^{-kx}\,dx = \left| \begin{matrix} t = -kx \\ dt = -k\,dx \end{matrix} \right| = -\int e^t \cdot \frac{t}{k}\,dt$$

⇒ VAŽAN PRIMJER:

$$\int_a^\infty \frac{dx}{x^\lambda} = \lim_{L\to\infty} \int_a^L x^{-\lambda}\,dx = \lim_{L\to\infty} \frac{x^{-\lambda+1}}{-\lambda+1} \Big|_a^L =$$

$a>0,\ \lambda\neq 1,\ \lambda\in\mathbb{R}$

$$= \frac{1}{1-\lambda}\left[\lim_{L\to\infty} \frac{1}{L^{\lambda-1}} - \frac{1}{a^{\lambda-1}}\right] =$$

$$= \begin{cases} \frac{1}{1-\lambda}\left(0 - \frac{1}{a^{\lambda-1}}\right), & \lambda > 1 \\ +\infty, & \lambda < 1 \end{cases} = \begin{cases} \frac{1}{\lambda-1}\cdot\frac{1}{a^{\lambda-1}}, & \lambda > 1 \\ +\infty, & \lambda < 1 \end{cases}$$

↳ dakle: $\int_a^\infty \frac{dx}{x^\lambda} = \begin{cases} \text{konvergira za } \lambda > 1 \\ \text{divergira za } \lambda \leq 1 \end{cases}$

PR: $\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{\substack{l\to-\infty \\ L\to\infty}} \int_{l}^{L} \frac{dx}{x^2+1} = \lim_{\substack{l\to-\infty \\ L\to\infty}} \operatorname{arctg} x \Big|_{l}^{L} =$

$= \lim_{\substack{l\to-\infty \\ L\to\infty}} \left(\operatorname{arctg} L - \operatorname{arctg} l\right) = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

↳ ovaj integral konvergira jer ima konačan rezultat

↳ KOREKTNIJI ZAPIS:

$$\int_{-\infty}^{0} \frac{dx}{x^2+1} + \int_{0}^{\infty} \frac{dx}{x^2+1} = \ldots\ldots = \pi$$

ZAD: $\int_0^{\infty} e^{-\sqrt{x}}\,dx = \left| \begin{array}{l} t=\sqrt{x},\ t>0 \\ t^2=x \\ 2t\,dt=dx \\ t(0)=0\ \ t(\infty)=\infty \end{array} \right| = 2\int_0^{\infty} e^{-t}\,t\,dt = \left| \begin{array}{ll} u=t & dv=e^{-t}dt \\ du=dt & v=-e^{-t} \end{array} \right| =$

$= 2\cdot\left[-te^{-t}\Big|_0^{\infty} + \int_0^{\infty} e^{-t}\,dt\right] = 2\left[-te^{-t}\Big|_0^{\infty} + \left(-e^{-t}\right)\Big|_0^{\infty}\right] = (*)$

$\lim_{t\to\infty}\left(te^{-t}\right) = \lim_{t\to\infty} \frac{t}{e^t} \overset{[L'H]}{=} \lim_{t\to\infty} \frac{1}{e^t} = 0$

$(*) = 2\cdot\left[0-0-(0-1)\right] = 2$

ZAD: $\displaystyle\int_0^\infty \frac{dx}{\sqrt{(x^2+1)^3}} = \left| \begin{array}{l} x = \operatorname{sh} t \\ dx = \operatorname{ch} t\, dt \\ t(0)=0 \quad t(\infty)=\infty \end{array} \right| = \int_0^\infty \frac{\operatorname{ch} t\, dt}{\sqrt{\operatorname{ch}^6 t}} = \int_0^\infty \frac{\operatorname{ch} t\, dt}{\operatorname{ch}^3 t} =$

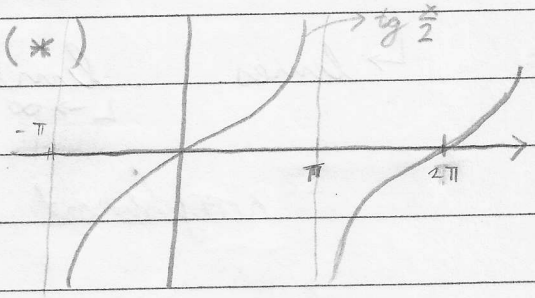
$$= \int_0^\infty \frac{dt}{\operatorname{ch}^2 t} = \operatorname{th} t \Big|_0^\infty = \operatorname{th}(\infty) - \operatorname{th}(0) = 1 - 0 = 1$$

ZAD: $\displaystyle\int_0^\infty \frac{x\,dx}{x^4+3x^2+3} = \left| \begin{array}{l} t = x^2 \\ dt = 2x\,dx \\ t(0)=0 \quad t(\infty)=\infty \end{array} \right| = \frac{1}{2}\int_0^\infty \frac{dt}{t^2+3t+3} = \frac{1}{2}\int_0^\infty \frac{dt}{\left(t+\frac{3}{2}\right)^2+\frac{3}{4}} =$

$$= \left| \begin{array}{l} u = t + \frac{3}{2} \\ du = dt \\ u(0)=\frac{3}{2} \quad u(\infty)=\infty \end{array} \right| = \frac{1}{2}\int_{\frac{3}{2}}^\infty \frac{du}{u^2+\frac{3}{4}} = \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\cdot\left[\lim_{u\to\infty} \operatorname{arctg}\left(\frac{2u}{\sqrt{3}}\right) - \operatorname{arctg}\left(\frac{2\cdot 3}{2\sqrt{3}}\right)\right]$$

$$= \frac{1}{\sqrt{3}}\left[\frac{\pi}{2} - \frac{\pi}{3}\right] = \frac{\pi}{6\sqrt{3}}$$

ZAD: $\displaystyle\int_0^{2\pi} \frac{dx}{2+\cos x} = \left| \begin{array}{l} t = \operatorname{tg}\frac{x}{2} \\ \cos x = \frac{1-t^2}{1+t} \\ dx = \frac{2\,dt}{1+t^2} \end{array} \right| = (*)$



$\displaystyle\int_0^{2\pi} \frac{dx}{2+\cos x} = \int_0^{\pi} \frac{dx}{2+\cos x} + \int_{\pi}^{2\pi} \frac{dx}{2+\cos x}$ vidimo da je funkcija prekidna te ako hočemo zadržati univerzalnu supstituciju, moramo početni integral rastaviti !

$$(*) = \int_0^\infty \frac{\frac{2\,dt}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} + \int_{-\infty}^0 \frac{\frac{2\,dt}{1+t}}{2+\frac{1-t^2}{1+t^2}} = \ldots\ldots\ldots = \frac{2\pi\sqrt{3}}{3}$$

↳ možda je bolja supstitucija: $\operatorname{ctg}\frac{x}{2}$

ZAD: $\int_{-\infty}^{\infty} \frac{x\,dx}{x^2+1} = \int_{-\infty}^{0} \frac{x\,dx}{x^2+1} + \int_{0}^{\infty} \frac{x\,dx}{x^2+1}$

↳ ako je jedan od integrala u zbroju divergentan, onda je i glavni integral divergentan pa ne treba drugog ni računat

$$\int_0^{\infty} \frac{x\,dx}{x^2+1} = \lim_{L\to\infty} \int_0^{L} \frac{x\,dx}{x^2+1} = \lim_{L\to\infty} \frac{1}{2}\ln|x^2+1|\Big|_0^{L} = \infty$$

⇒ NAPOMENA:

$$\lim_{L\to\infty} \int_{-L}^{L} \frac{x\,dx}{x^2-1} = 0$$

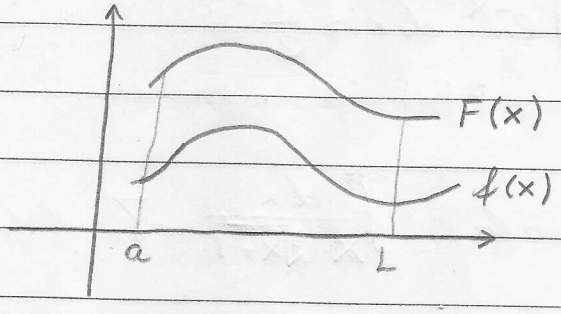
$L > 0$

POGREŠNO!

↳ limes $\lim_{L\to\infty} \int_{-L}^{L} f(x)\,dx$ se zove glavna vrijednost integrala $\int_{-\infty}^{\infty} f(x)\,dx$

ISPITIVANJE KONVERGENCIJE NEPRAVIH INTEGRALA PRVE VRSTE

⇒ 1. KRITERIJ:



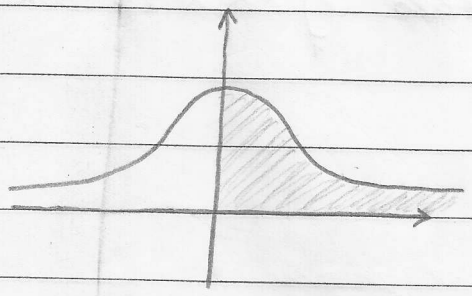
$$0 \le f(x) \le F(x), \quad \forall x \in [a, \infty\rangle$$

$$\int_a^\infty f(x)\,dx \le \int_a^\infty F(x)\,dx$$

↳ ako $\int_a^\infty F(x)\,dx$ konvergira, onda konvergira i $\int_a^\infty f(x)\,dx$ ▽

↳ pogledaj PR 6 ; PR 8 ▽

PR: Ispitaj konvergenciju integrala $\int_0^\infty e^{-x^2}dx$?



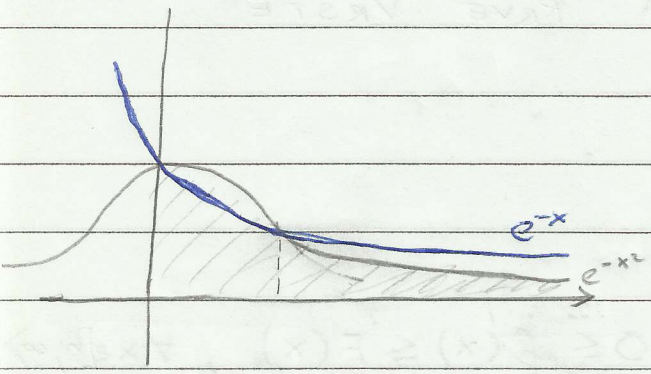
↳ želimo $f(x) = e^{-x^2}$ ograničiti odozgo s nekom funkcijom koji znam integrirat

$$\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{ovaj je sigurno konvergentan}} + \int_1^\infty e^{-x^2}dx$$

↳ zato treba samo pokorat konvergenciju $\int_1^\infty e^{-x^2}dx$

OKRENI

$e^{-x^2} \leq e^{-x}, \quad \forall x \in [1, \infty\rangle$



$$\int_1^\infty e^{-x^2} dx \leq \int_1^\infty e^{-x} dx$$

$$\int_1^\infty e^{-x^2} dx \leq \frac{1}{e}$$

ZAD: Ispitaj konvergenciju integrala $\int_1^\infty \frac{dx}{x \cdot \sqrt[3]{x+1}}$?

$$f(x) = \frac{1}{x \cdot \sqrt[3]{x+1}} \leq \frac{1}{x \cdot \sqrt[3]{x}} = \frac{1}{x^{\frac{4}{3}}}$$

↳ znamo iz prijašnjih primjera da:

$\int_1^\infty \frac{dx}{x^{\frac{4}{3}}}$ konvergira jer $\int_a^\infty \frac{dx}{x^2} = \begin{cases} \text{DVG. } 2 \leq \\ \text{KVG. } 2 > \end{cases}$

↳ zaključujemo da je $\int_1^\infty \frac{dx}{x \cdot \sqrt[3]{x+1}}$ konvergentna

ZAD: Ispitaj konvergenciju integrala $\int_1^\infty \cos x \cdot 3^{-x^2} dx$

$$3^{-x^2} = e^{\ln 3^{-x^2}} = e^{-x^2 \cdot \ln 3}$$

$$f(x) = \cos x \cdot 3^{-x^2}$$

$$\left|\cos x \cdot 3^{-x^2}\right| = \underbrace{|\cos x|}_{\leq 1} \cdot 3^{-x^2} \leq 3^{-x^2}$$

↳ znači treba samo ograničiti funkciju 3^{-x^2}

$$\frac{1}{3^{-x^2}} \leq \frac{1}{3^{-x}} \quad , \quad \forall x \in [1, \infty\rangle$$

$$\int_1^\infty 3^{-x}\,dx = \int_1^\infty e^{-x\ln 3}\,dx = \left| \begin{matrix} u = -x\ln 3 \\ du = -\ln 3\,dx \end{matrix} \right| =$$

$$= \int_?^? e^u \cdot \left(-\frac{1}{\ln 3}\right) du = -\frac{1}{\ln 3} \cdot e^{-x\ln 3} \Big|_1^\infty = -\frac{1}{\ln 3}\left(\lim_{x\to\infty} e^{-x\ln 3} - e^{-\ln 3}\right)$$

$$= \frac{1}{\ln 3} \cdot \frac{1}{3} = \frac{1}{3\ln 3}$$

↳ pokazali smo da $\int_1^\infty |f(x)|\,dx$ konvergira

↳ znamo da je integral od $f(x)$ sigurno manji il jednak od integrala $|f(x)|$ pa je time konvergentnost integrala:

$$\int_1^\infty \cos x \cdot 3^{-x^2}\,dx \quad \text{dokazana !}$$

→ **NAPOMENA:** iz ovog uvjeta:

$$0 \leq f(x) \leq F(x) \quad , \forall x \in [a, \infty\rangle$$

$$\int_a^\infty f(x)\,dx \leq \int_a^\infty F(x)\,dx$$

↳ ne možemo ništa zaključiti o konvergenciji $\int_a^\infty f(x)\,dx$, ako $\int_a^\infty F(x)\,dx$ divergira, ali ako $\int_a^\infty f(x)\,dx$ divergira, onda sigurno divergira $\int_a^\infty f(x)\,dx$

⇒ **2. KRITERIJ**: ispitivanje pomoću limesa

$$F(x),\quad f(x) \geq 0,\quad \forall x \in [a, \infty\rangle$$

↳ ako je: $\lim\limits_{x\to\infty} \frac{f(x)}{F(x)} = 1$, tada se integral $\int_a^\infty f(x)\,dx$ i $\int_a^\infty F(x)\,dx$ ponašaju jednako (oba konvergiraju il oba divergiraju)

PR: Ispitaj konvergenciju integrala $\int_1^\infty \frac{x\,dx}{3x^4+5x^2+1}$ ∇

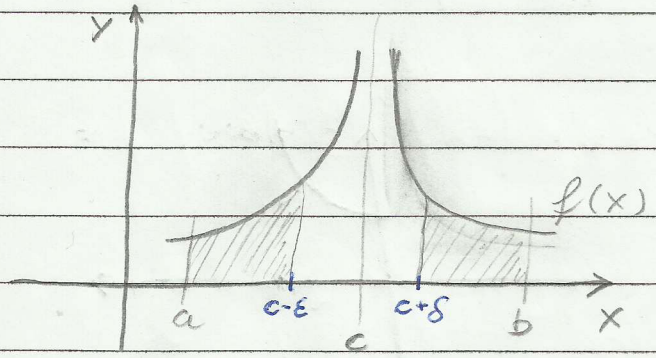
$$\lim_{x\to\infty} \frac{\frac{x}{3x^4+5x^2+1}}{\frac{1}{3x^3}} = 1$$

$\int_1^\infty \frac{1}{3x^3}\,dx$ ⇒ ovaj integral konvergira, pa konvergira i polazni integral

(DZ) ⇒ ispitaj konvergenciju integrala:

$$\int_3^\infty x^2 \ln x$$

NEPRAVI INTEGRALI DRUGE VRSTE



$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0^+} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta \to 0^+} \int_{c+\delta}^b f(x)\,dx$$

⇒ polazni integral konvergira ako oba limesa postoje

⇒ ako jedan od limesa nije konačan ⇒ onda integral divergira

PR: Izračunaj $\int_{-1}^{1} \frac{dx}{x^4}$ ∇

↳ ne možemo primjeniti Newton-Leibnizovu formulu jer je funkcija prekidna

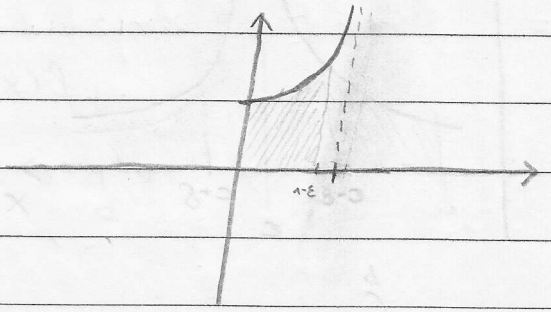
$$\int_{-1}^{1} \frac{dx}{x^4} = \lim_{\varepsilon \to 0^+} \int_{-1}^{-\varepsilon} \frac{dx}{x^4} + \lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{dx}{x^4} = I_1 + I_2$$

$$I_2 = \lim_{\delta \to 0^+} \int_{\delta}^{1} \frac{dx}{x^4} = -\frac{1}{3} \lim_{\delta \to 0} \left(1 - \frac{1}{\delta^3}\right) = \infty$$

↳ integral divergira ∇

PRS Važan primjer ▽ ! ▽ ! ▽

Izračunaj $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ ▽



$$f(x) = \frac{dx}{\sqrt{1-x^2}}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon \to 0^+} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x^2}} = \lim_{\varepsilon \to 0^+} \arcsin x \Big|_0^{1-\varepsilon} =$$

$$= \lim_{\varepsilon \to 0^+} \left[\arcsin(1-\varepsilon) - 0 \right] = \frac{\pi}{2}$$

ISPITIVANJE KONVERGENCIJE NEPRAVIH INTEGRALA DRUGE VRSTE

⇒ većinom koristimo 2. kriterij:

POMOĆU LIMESA

↳ konvergenciju ispitujem gotovo isključivo pomoću limesa

⇒ KRITERIJ „LIMESA":

↳ ako je:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = 1$$, onda ili oba integrala konvergiraju ili oba divergiraju

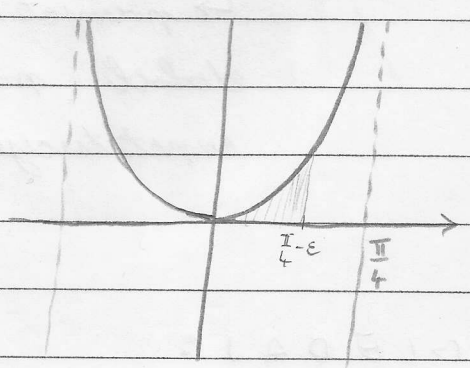
↳ napomena: funkcija ima prekid u točki „c"

ZAD: (Slično zadatku 5 iz 13. DZ)

Izračunaj:

$$\int_0^{\frac{\pi}{4}} tg^2(2x)\,dx =$$

$$= \lim_{\varepsilon \to 0^+} \int_0^{\frac{\pi}{4}-\varepsilon} tg^2(2x)\,dx = (*)$$



↳ riješimo neodređeni integral:

$$\int tg^2(2x)\,dx = \int \frac{\sin^2 2x}{\cos^2 2x}\,dx = \int \frac{1-\cos^2 2x}{\cos^2 2x}\,dx = \int \frac{dx}{\cos^2(2x)} - \int dx =$$

OKRENI →

$$= \left| \begin{matrix} u = 2x \\ du = 2dx \end{matrix} \right| = \frac{1}{2} \int \frac{du}{\cos^2 u} = \frac{1}{2} \operatorname{tg}(2x) + C$$

↳ vratimo se u početni integral

$$(*) = \lim_{\varepsilon \to 0^+} \int_0^{\frac{\pi}{4}-\varepsilon} \operatorname{tg}^2(2x)\, dx = \lim_{\varepsilon \to 0^+} \left(-x \Big|_0^{\frac{\pi}{4}-\varepsilon} + \frac{1}{2} \operatorname{tg}(2x) \Big|_0^{\frac{\pi}{4}-\varepsilon} \right) =$$

$$= \lim_{\varepsilon \to 0^+} \left(-\left(\frac{\pi}{4} - \varepsilon - 0\right) + \frac{1}{2} \operatorname{tg}\left(\frac{\pi}{2} - 2\varepsilon\right) - 0 \right) = \infty$$

(DZ rad 4. b))

⇒

⇒ NAPOMENA:

$$\text{PR: } \int_0^2 \frac{x^2 dx}{\sqrt{4-x^2}} = \left| \begin{matrix} x = 2\sin t \\ \vdots \end{matrix} \right| = \ldots = 8 \int_0^{\frac{\pi}{2}} \sin^3 t\, dt$$

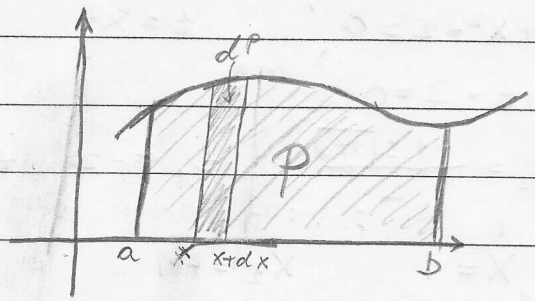
$$\lim_{x \to 2^-} \frac{x^3}{\sqrt{4-x^2}} = \infty$$

↳ prihvatili da smo iz nepravog integrala dobili pravi primjenom odredene supstitucije

PRIMJENA ODREĐENOG INTEGRALA

① Površina ravninskog lika

a)

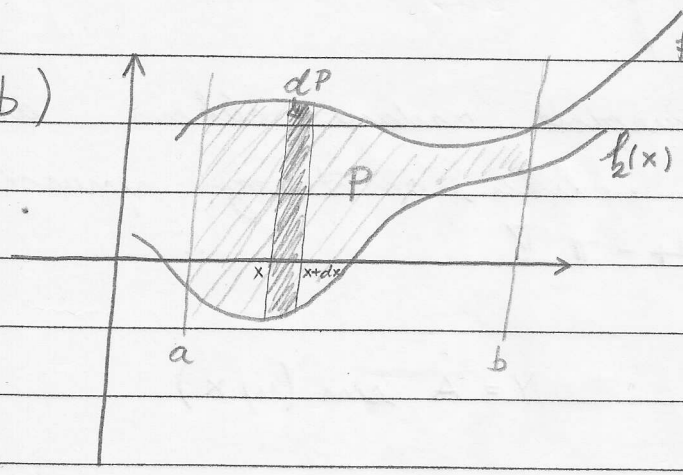


$f(x) \geq 0$

$$P = \int_a^b f(x)\,dx$$

$$dP = f(x)\,dx$$

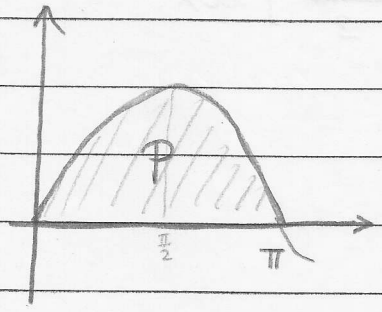
b)



$$dP = (f_1(x) - f_2(x))\,dx$$

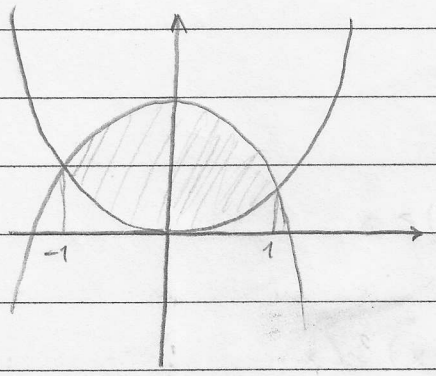
$$P = \int_a^b (f_1(x) - f_2(x))\,dx$$

ZADO: Nadi površinu lika između sinusoide $y = \sin x$ i osi $y = 0$ na intervalu $x \in [0, \pi]$?



$$P = \int_0^{\pi} \sin x\,dx = -\cos x \Big|_0^{\pi} = 2$$

ZAD: Nadi površinu lika omeđenog krivuljama $y = x^4$ i $y = 2 - x^2$ ∇



$$2 - x^2 = x^4$$

$$x^4 + x^2 - 2 = 0 \qquad t = x^2$$

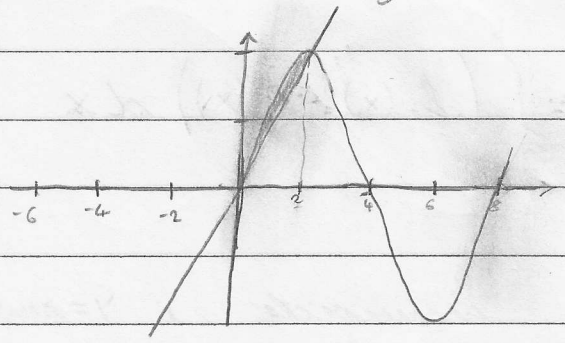
$$t^2 + t - 2 = 0$$

$$t_{1,2} = \frac{-1 \pm \sqrt{1+8}}{2} \qquad \cancel{t_1 = -2} \qquad t_2 = 1$$

$$x = 1 \qquad x_2 = -1$$

$$P = \int_{-1}^{1} [(2 - x^2) - x^4]\,dx = 2\int_{0}^{1} [2 - x^2 - x^4]\,dx = \ldots = \frac{44}{15}$$

ZAD: Nadi amplitudu sinusoide zadane slikom ako je površina nacrtanog lika omeđenog sinusoidom i pravcem jednaka $4 - \pi$ ∇



$$y = A \sin(\omega x)$$

$$T = 8 = \frac{2\pi}{\omega} \Rightarrow \omega = \frac{\pi}{4}$$

$$4 - \pi = \int_{0}^{2} \left[A \sin\left(\frac{\pi}{4} x\right) - \frac{A x}{2}\right] dx$$

$$\vdots$$

$$A = \pi$$

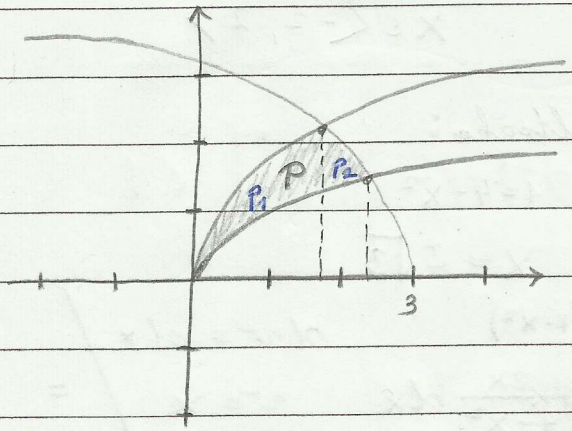
Zad: Izračunaj površinu lika omeđenog krivuljama:

$y = \sqrt{x}$

$y = 2\sqrt{x}$

$y = \sqrt{6-2x}$ $\Rightarrow$ $6-2x \geq 0$

$x \leq 3$



$\Rightarrow$ površinu podijelimo na dva dijela, na laka računamo integrale

$\Rightarrow$ točke presjeka:

$2\sqrt{x} = \sqrt{6-2x} \;/^2$

$4x = 6-2x$

$x = 1$

$\sqrt{x} = \sqrt{6-2x} \;/^2$

$x = 6-2x$

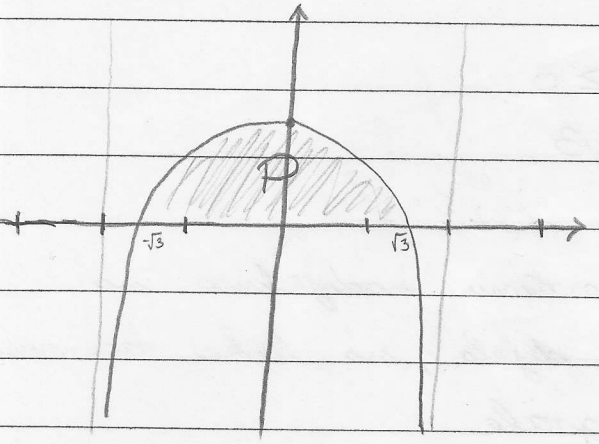
$x = 2$

$$P = P_1 + P_2 = \int_0^1 \left(2\sqrt{x} - \sqrt{x}\right) dx + \int_1^2 \left(\sqrt{6-2x} - \sqrt{x}\right) dx$$

$\vdots$

$P = 4 - 2\sqrt{2}$

ZAD: Izračunaj površinu lika omeđenog grafom krivulje $y = \ln(4 - x^2)$ i osi x!



$4 - x^2 > 0$

$x \in \langle -2, 2 \rangle$

↳ multočka:

$1 = 4 - x^2$

$x = \pm\sqrt{3}$

$$P = 2 \cdot \int_0^{\sqrt{3}} \ln(4 - x^2)\,dx = \left| \begin{array}{ll} u = \ln(4-x^2) & dv = dx \\ du = -\frac{2x}{4-x^2}\,dx & v = x \end{array} \right| =$$

$$= 2 \left(x \ln(4 - x^2) \Big|_0^{\sqrt{3}} - \int_0^{\sqrt{3}} -\frac{2x^2}{4 - x^2}\,dx \right)$$

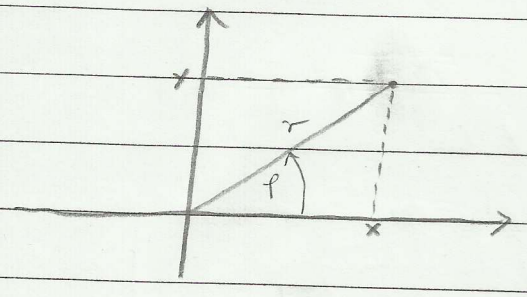
$$= 2 \left(x \ln(4 - x^2) \Big|_0^{\sqrt{3}} + 2 \int_0^{\sqrt{3}} \frac{x^2}{4 - x^2}\,dx \right)$$

⋮

jednostavna integracija

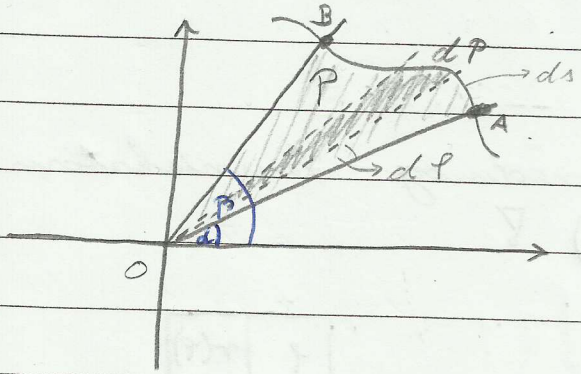
$$P = 4 \ln\left(7 + 2\sqrt{3}\right) - 4\sqrt{3}$$

POVRŠINA SEKTORA I POLARNE KOORDINATE



$$x = r\cos\varphi$$
$$y = r\sin\varphi$$

⇒ POVRŠINA SEKTORA:



⇒ luk OAB omeđen krivuljom $r = r(\varphi)$ i zrakama $\varphi = \alpha$; $\varphi = \beta$ nove se SEKTOR

↳ jer je $d\varphi$ neizmjerno mali, možemo luk krivulje zamijeniti s lukom kružnice

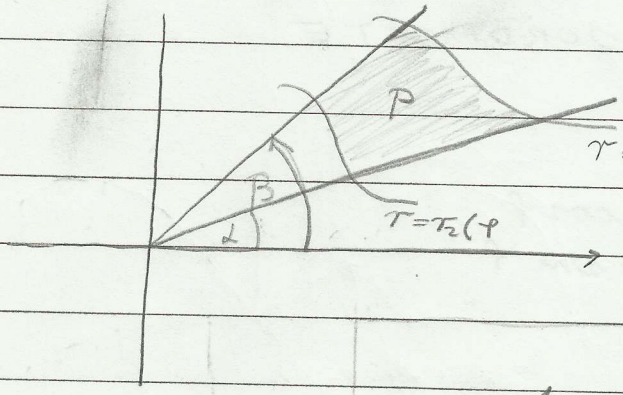
$$dP = ds \cdot \frac{r}{2} = r\,d\varphi \cdot \frac{r}{2}$$

$$dP = \frac{1}{2} r^2 d\varphi$$

$$P = \int dP = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\varphi$$

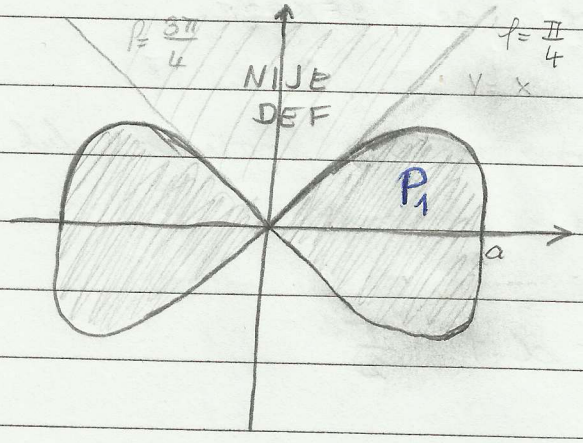
$$\boxed{P = \frac{1}{2} \int_{\alpha}^{\beta} r^2(\varphi)\, d\varphi}$$

⇒ nešto općenitije: (ISPIT ∇)



$$P = \frac{1}{2}\int_{\alpha}^{\beta}\left[r_1^2(\varphi) - r_2^2(\varphi)\right]d\varphi$$

ZAD: Nađi površinu lika omeđenog lemniskatom jednadžbe $r^2 = a^2\cos(2\varphi)$ ∇



$r(\varphi) = r(-\varphi)$
↳ krivulja simetrična s obzirom na x os

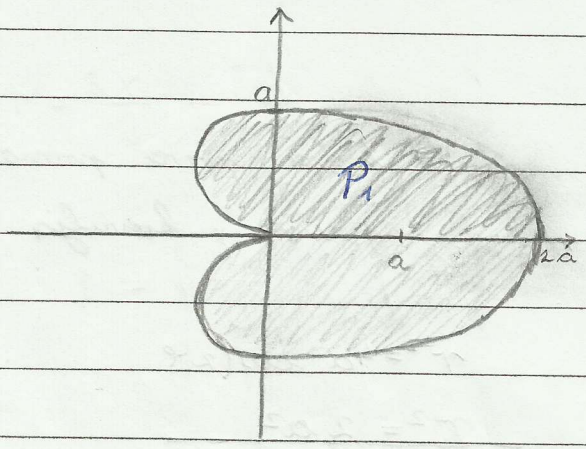
φ	$r(\varphi)$
$\emptyset$	a
$\frac{\pi}{4}$	$\emptyset$
$\frac{3\pi}{4}$	$\emptyset$
π	a

$P = 4P_1$

$$P = 4\cdot\frac{1}{2}\int_0^{\frac{\pi}{4}} a^2\cos(2\varphi)\,d\varphi = 2a^2\cdot\frac{\sin(2\varphi)}{2}\Big|_0^{\frac{\pi}{4}} =$$

$$P = a^2(1-0) = a^2$$

ZAD: Izračunaj površinu lika omeđenog kardioidom $r = a(1 + \cos\varphi)$



	φ	$r(\varphi)$
$r(\varphi) = r(-\varphi)$	0	$2a$
↳ krivulja	$\frac{\pi}{3}$	$\frac{3}{2}a$
unutrašnja s	$\frac{\pi}{2}$	a
- dirnom na x-os	π	0

$$P = 2P_1 = 2 \cdot \frac{1}{2} \int_0^{\pi} r^2(\varphi)\, d\varphi$$

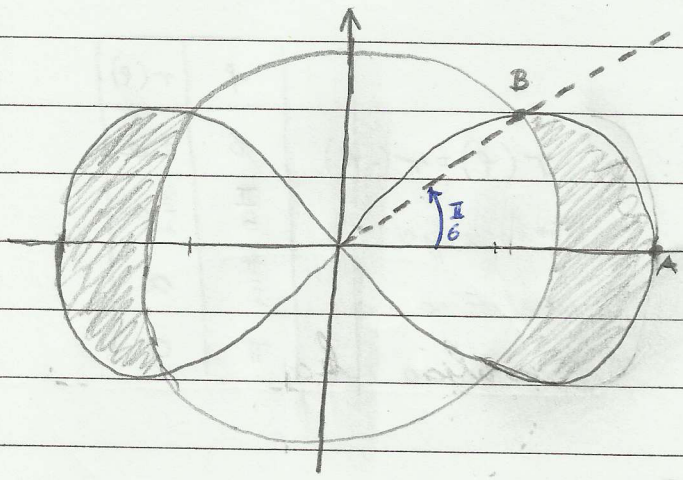
$$P = \int_0^{\pi} \left[a(1+\cos\varphi)\right]^2 d\varphi = a^2 \int_0^{\pi} (1 + 2\cos\varphi + \cos^2\varphi)\, d\varphi$$

$$P = a^2 \int_0^{\pi} \left(1 + 2\cos\varphi + \frac{1+\cos(2\varphi)}{2}\right) d\varphi =$$

$$P = a^2 \int_0^{\pi} \left(\frac{3}{2} + \underbrace{2\cos\varphi}_{0} + \underbrace{\frac{\cos(2\varphi)}{2}}_{0}\right) d\varphi$$

$$P = \frac{3}{2} a^2 \pi$$

ZAD: Izračunaj površinu lika unutar lemniskate $r^2 = 4a^2\cos(2\varphi)$, a izvan kružnice $r = a\sqrt{2}$!



⇒ treba prvo pronaći sjecišta krivulja

$r^2 = 4a^2\cos(2\varphi$

$r^2 = 2a^2$

⋮

$\cos 2\varphi = \frac{1}{2}$

$\varphi = \pm \frac{\pi}{6}$

$$\frac{1}{4}P = \frac{1}{2}\int_0^{\frac{\pi}{6}} \left[r_1^2(\varphi) - r_2^2(\varphi)\right] d\varphi \quad / \cdot 4$$

$$P = 2\int_0^{\frac{\pi}{6}} \left[4a^2\cos 2\varphi - 2a^2\right] d\varphi$$

$$P = 2\cdot\left(2a^2 \sin 2\varphi - 2a^2\varphi\right)\Big|_0^{\frac{\pi}{6}}$$

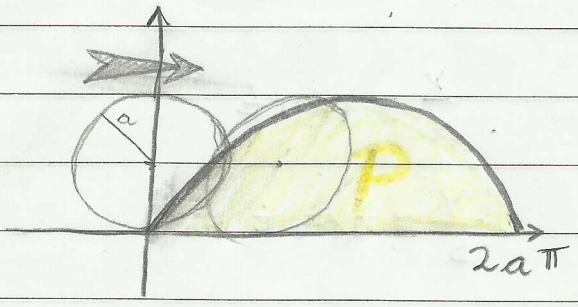
⋮

$$P = \frac{2}{3}\left(3\sqrt{3} - \pi\right)\cdot a^2$$

⇒ POVRŠINA LIKA OMEĐENOG KRIVULJOM KOJA JE ZADANA U PARAMETARSKOM OBLIKU:

$$x = x(t)$$
$$y = y(t) \qquad t \in [t_1, t_2]$$

PR: Cikloida ∇



→ krivulja koja nastaje kotrljanjem kružnice po osi apscisa

$$x = a(t - \sin t) \qquad t \in [0, 2\pi]$$
$$y = a(1 - \cos t)$$

$$P = \int_{x_1}^{x_2} y(x)\,dx$$

$$P = \int_0^{2\pi} \underbrace{a(1-\cos t)}_{y(t)} \cdot \underbrace{a(1-\cos t)\,dt}_{\dot{x}(t)\,dt}$$

$$P = a^2 \int_0^{2\pi} (1-\cos t)^2 dt = a^2 \int_0^{2\pi} (1 - 2\cos t + \cos^2 t)\,dt$$

$$P = a^2 \left[t \Big|_0^{2\pi} - 2\sin t \Big|_0^{2\pi} \right] + a^2 \int_0^{2\pi} \frac{1+\cos 2t}{2}\,dt$$

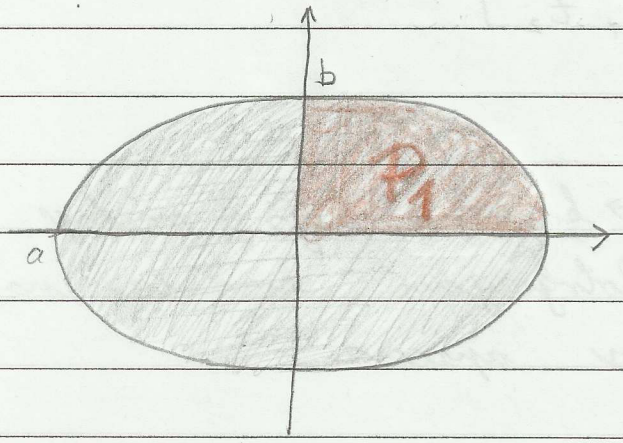
$$P = a^2 \left[2\pi - 2\cdot 0 + \frac{1}{2} \left(t \Big|_0^{2\pi} + \frac{1}{2}\sin(2t) \Big|_0^{2\pi} \right) \right]$$

$$P = 3a^2\pi$$

⇒ poglejdaj primer Astroide (PR 25 ⇒ str 20)

↳ može se rješiti bez rekurzivnih formula

Zad: Izračunaj površinu elipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ▽



$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$$

$$\frac{x}{a} = \cos t$$

$$\frac{y}{b} = \sin t$$

$$x = a\cos t$$

$$y = b\sin t$$

↳ od kud do kud se mijenja „t":

$t = 0$: $x = a$, $y = 0$

$t = \frac{\pi}{4}$: $x = \frac{a\sqrt{2}}{2}$, $y = \frac{b\sqrt{2}}{2}$

⋮

$$t \in [0, 2\pi\rangle$$

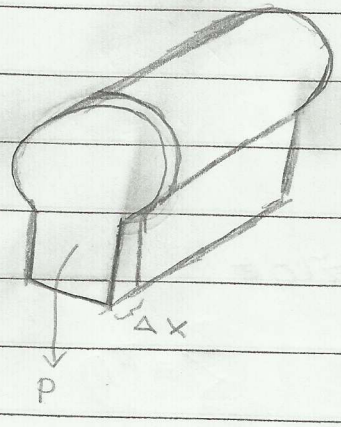
↳ s obzirom da nam x pada s porastom parametra „t", integriramo od $\frac{\pi}{2}$ do 0 kako nam površina ne bi bila negativna:

$$P_1 = \int_{\frac{\pi}{2}}^{0} b\sin t \cdot (-a)\sin t\, dt = ab \int_{0}^{\frac{\pi}{2}} \sin^2 t\, dt$$

⋮

$$P_1 = \frac{\pi ab}{4} \Rightarrow P = \pi ab$$

IZRAČUNAVANJE VOLUMENA



⇒ računanje volumena integralnog kruha

$$\Delta V \approx P \Delta x$$

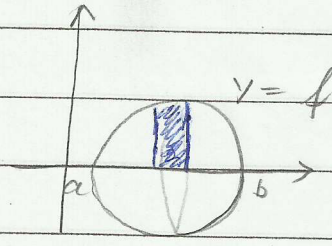
"na limesu"

$$dV = P(x)\,dx$$

$$V = \int dV = \int P(x)\,dx$$

↳ pogledaj primjer u knjižici

⇒ VOLUMEN ROTACIJSKOG TIJELA:



πy^2 je površina

dx je debljina

$$dV = \pi y^2\,dx$$

$$V = \pi \int_a^b y^2(x)\,dx$$

MATRICE

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 & -1 \\ 2 & 1 & \pi \end{bmatrix}, \quad C = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad D = [1, 2, 3]$$

$A \in R^{2\times 2}$ $B \in R^{2\times 3}$ $C \in R^{3\times 1}$ $D \in R^{1\times 3}$

⇒ matrica je „pravokutna tablica" realnih brojeva
↳ predočavamo je kao dvodimenzionalno polje

⇒ OPĆENITA MATRICA:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

$$A \in R^{m\times n}$$

$$(A)_{ij} = a_{ij}$$

i ⇒ indeks retka

j ⇒ indeks stupca

⇒ JEDNAKOST MATRICA:

↳ matrice A i B su jednake ako su istog tipa i ako vrijedi $a_{ij} = b_{ij}$ $\forall i, j$

⇒ TIPOVI MATRICA:

a) nul - matrica:

$$a_{ij} = 0 \quad \forall i, j$$

b) dijagonalna matrica:

↳ definirana za kvadratne matrice

↳ svi elementi izvan glavne dijagonale su jednaki nuli

c) jedinična matrica:

↳ dijagonalna matrica čiji su elementi na dijagonali jednaki 1

↳ oznaka: $I = (\delta_{ij}) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$

δ ← Kroneckerov „delta" simbol

d) trokutaste matrice:

↳ definirana za kvadratne matrice

↳ elementi ispod ili iznad glavne dijagonale su jednaki nuli

e) transponirana matrica matrice A

↳ matrica koju dobijemo iz matrice A zamjenom redaka i stupaca

↳ oznaka: $A \longrightarrow B = A^T$

PR:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 2 & 3 \\ -1 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & -1 \\ 0 & 2 \\ 1 & 4 \end{bmatrix} \qquad B^T = \begin{bmatrix} 3 & 0 & 1 \\ -1 & 2 & 4 \end{bmatrix}$$

↳ primjet: $(A^T)_{ij} = (A)_{ji} \quad \forall i, j$

↳ primjet: $(A^T)^T = A$

f) simetrična matrica (VAŽNA)

↳ matrica koja je jednaka svojoj transponiranoj matrici

$$A^T = A$$

↳ definirana za kvadratne matrice

g) vektor kao matrica

1) vektor redak: $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ 1×3

2) vektor stupac: $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ 3×1

PR: $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}^T = \begin{bmatrix} 3 & 2 & 1 \end{bmatrix}$

OPERACIJE S MATRICAMA

$\mathcal{M}_{m,n}$ ⇒ skup svih realnih matrica tipa $m \times n$

↳ u skupu definiramo:

- zbrajanje matrica
- množenje matrica skalarom

⇒ zbrajanje matrica ⇒ zbrajamo odgovarajuće elemente matrica

$$(A + B)_{ij} = (A)_{ij} + (B)_{ij}$$

⇒ množenje matrica skalarom ⇒ množenje svakog elementa matrice sa skalarom

$$\lambda (A)_{ij} = (\lambda A)_{ij}$$

$\lambda \in R$

ALGEBRA MATRICA

⇒ MNOŽENJE:

↳ da bi umnožak matrica bio definiran, one moraju biti „ulančane" jednom svojom dimenzijom

PR: umnožak vektor-redak s vektor-stupcem

$$a = [a_1\ a_2 \dots a_m] \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$1 \times m$ $\qquad$ $m \times 1$

ulančane dimenzije

$$a \cdot b = [a_1\ a_2 \dots a_m] \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} = a_1 b_1 + a_2 b_2 + \dots + a_m b_m =$$

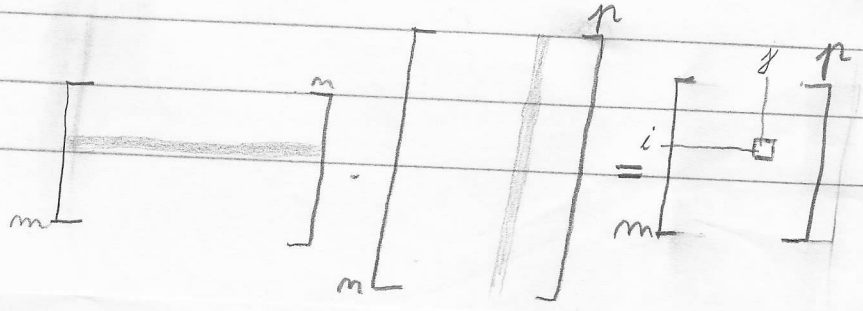
$$= \sum_{k=1}^{m} a_k b_k \quad \rightarrow \text{realan broj}$$

↳ skalarni produkt vektora „a" i „b" u R^m

↳ općenito množenje matrica:

$A = (a_{ij})$ tipa $m \times n$

$B = (b_{ij})$ tipa $n \times p$ $\rightarrow$ $AB = (c_{ij})$ tipa $m \times p$



$$(AB)_{ij} = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{im} \end{bmatrix} \begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{ij} \end{bmatrix} = a_{i1} \cdot b_{1j} + a_{i2} b_{2j} + \dots + a_{im} b_{mj}$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

PR:

$$A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 2 & 1 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix}$$

↳ Pomnoži matrice ▽!

$4 \times 2 \qquad 2 \times 3$

4×3

$$A \cdot B = \begin{bmatrix} 1\cdot3+2\cdot1 & 1\cdot1+2\cdot(-1) & 1\cdot2+2\cdot0 \\ -1\cdot3+1\cdot1 & -1\cdot1+1\cdot(-1) & -1\cdot2+1\cdot0 \\ 2\cdot3+1\cdot1 & 2\cdot1+1\cdot(-1) & 2\cdot2+1\cdot0 \\ 1\cdot3+0\cdot1 & 1\cdot1+0\cdot(-1) & 1\cdot2+0\cdot0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ -2 & -2 & -2 \\ 7 & 1 & 4 \\ 3 & 1 & 2 \end{bmatrix}$$

⇒ NAPOMENA: ne vrijedi komutativnost množenja

$$AB \neq BA$$

PR:

$$\underset{A}{\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix}} \quad \underset{B}{\begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AB = 0, \quad \text{al} \quad A \neq 0$$

$$B \neq 0$$

↑ nije nula, već nul-matrica

PR: $x = [3 \quad 1 \quad 2]$, $y = [2 \quad -1 \quad 1]$

$$x^T y = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} [2 \quad -1 \quad 1] = \begin{bmatrix} 6 & -3 & 3 \\ 2 & -1 & 1 \\ 4 & -2 & 2 \end{bmatrix}$$

$$y\, x^T = [2 \quad -1 \quad 1] \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 7$$

skalarni produkt vektora x, y

MATRICE I LINEARNO PRESLIKAVANJE

$A \in R^{m \times n}$, $x \in R^{n}$ (vektor stupac)

$$y = Ax \in R^{m \times 1}$$ (vektor stupac)

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$$

$$y_1 = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n$$
$$y_2 = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n$$
$$\vdots$$
$$y_m = a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n$$

$\Rightarrow A: R^n \to R^m$ je linearno preslikavanje

$$A(\alpha u + \beta v) = \alpha A(u) + \beta A(v)$$

$\alpha, \beta \in R$; $u, v \in R^n$

↳ BITNO: kompoziciji linearnih preslikavanja odgovara umnožak pripadnih matrica

OKRENI →

⇒ zaista, pogledajmo:

$$B \in R^{2\times 2}; \quad f: R^2 \to R^2$$

$$B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \qquad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$y_1 = b_{11} x_1 + b_{12} x_2$$

$$y_2 = b_{21} x_1 + b_{22} x_2$$

- $A \in R^{2\times 2} \qquad g: R^2 \to R^2$

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z = Ay = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$z_1 = a_{11} y_1 + a_{12} y_2$$

$$z_2 = a_{21} y_1 + a_{22} y_2 \qquad \rightarrow \text{umjesto „}y\text{" odozgo}$$

$$z_1 = a_{11}(b_{11} x_1 + b_{12} x_2) + a_{12}(b_{21} x_1 + b_{22} x_2)$$

$$z_2 = a_{21}(b_{11} x_1 + b_{12} x_2) + a_{22}(b_{21} x_1 + b_{22} x_2)$$

$$z_1 = x_1(a_{11} b_{11} + a_{12} b_{21}) + x_2(a_{11} b_{12} + a_{12} b_{22})$$

$$z_2 = x_1(a_{21} b_{11} + a_{22} b_{21}) + x_2(a_{21} b_{12} + a_{22} b_{22})$$

SVOJSTVA MATRIČNOG MNOŽENJA

ASOCIJATIVNOST:

(malo teže)

$$(AB)\,C = A\,(BC)$$

$A = (a_{ij})$ tipa $m \times n$
$B = (b_{ij})$ tipa $n \times p$ } AB je tipa $m \times p$
$C = (c_{ij})$ tipa $p \times r$ } BC je tipa $n \times r$

$$\left[(AB)\,C\right]_{il} = \sum_{k=1}^{p} (AB)_{ik}\, c_{ik}$$

$$\left[A\,(BC)\right]_{il} = \sum_{j=1}^{n} a_{ij}\, (BC)_{jl}$$

$$(AB)_{ik} = \sum_{j=1}^{m} a_{ij}\, b_{jk} \quad , \quad (BC)_{jl} = \sum_{k=1}^{p} b_{jk}\, c_{kl}$$

↳ krenimo od $\left[(AB)\,C\right]_{il}$:

$$\left[(AB)\,C\right]_{il} = \sum_{k=1}^{p} (AB)_{ik}\, c_{kl} = \sum_{k=1}^{p} \left(\sum_{j=1}^{m} a_{ij} b_{jk} \right) c_{kl} =$$

$$= \sum_{j=1}^{m} a_{ij} \left(\sum_{k=1}^{p} b_{jk}\, c_{kl} \right)$$

DISTRIBUTIVNOST:

$$(A+B)C = AC + BC$$ (poglejmo dokaz v knjigi)

$A = (a_{ij})$ tipa $m \times n$

$B = (b_{ij})$ tipa $m \times n$

$C = (c_{jk})$ tipa $n \times p$

$$\left[(A+B)\cdot C\right]_{ik} = \sum_{j=1}^{n} (A+B)_{ij}\, c_{jk} = \sum_{j=1}^{n} (a_{ij} + b_{ij})\, c_{jk} =$$

$$= \sum_{j=1}^{n} a_{ij} c_{jk} + \sum_{j=1}^{n} b_{ij} c_{jk} =$$

$$= (AC)_{ik} + (BC)_{ik} =$$

$$= \left[AC + BC\right]_{ik}$$

UMNOŽAK S JEDINIČNOM MATRICOM:

$$A \in \mathcal{M}_n \quad , \quad I \in \mathcal{M}_n$$

$$A \cdot I = I \cdot A = A \qquad \text{(jednostavan dokaz)}$$

TRANSPONIRANJE: (bitno!)

$$(AB)^T = B^T A^T$$

↳ $A \Rightarrow$ tipa $m \times n$ / $A^T \Rightarrow$ tipa $n \times m$

↳ $B \Rightarrow$ tipa $n \times p$ / $B^T \Rightarrow$ tipa $p \times n$

$AB \Rightarrow$ tipa $m \times p$ / $(AB)^T \Rightarrow$ tipa $p \times m$

$B^T A^T$ je kojeg tipa?

↳ pa, dokažimo!

$$\left[(AB)^T\right]_{ik} = (AB)_{ki} = \sum_{j=1}^{n} a_{kj} b_{ji} = \sum_{j=1}^{n} b_{ji} a_{kj} =$$

$$= \sum_{j=1}^{n} (B^T)_{ij} (A^T)_{jk} = \left(B^T A^T\right)_{ik}$$

MATRIČNI POLINOM:

$$f(x) = \lambda_p x^p + \lambda_{p-1} x^{p-1} + \ldots + \lambda_1 x + \lambda_0$$

$$f(A) = \lambda_p A^p + \lambda_{p-1} x^{p-1} + \ldots + \lambda_1 A + \lambda_0 I$$

ZAD: Neka su A, B simetrične matrice. Pokaži da je i $AB+BA$ simetrična matrica.

$A = A^T$, $B = B^T$

↳ pokaži:

$$(AB+BA)^T = AB+BA$$

$$(AB+BA)^T = (AB)^T + (BA)^T = \quad \left(\Rightarrow \text{napomena: } (A+B)^T = A^T + B^T\right)$$

$$= B^T A^T + A^T B^T = BA + AB = AB + BA$$

ZAD: Neka su A, B kvadratne matrice. Pokaži da one komutiraju ako i samo ako je:

$$(A+B)^2 = A^2 + 2AB + B^2$$

① dokazujemo „udesno" ($\Rightarrow$)

$AB = BA$ (pretpostavka)

↳ pokaži: $(A+B)^2 = A^2 + 2AB + B^2$

$$(A+B)^2 = (A+B)(A+B) = (A+B)A + (A+B)B =$$

$$= A^2 + BA + AB + B^2 = A^2 + 2AB + B^2$$

② dokazujemo „ulijevo" ($\Leftarrow$)

$(A+B)^2 = A^2 + 2AB + B^2$ (pretpostavka)

↳ pokaži: $AB = BA$

⋮

ZAD: Neka je A kvadratna matrica. Pokaži da je $A \cdot A^T$ simetrična matrica.

$$(AA^T)^T = (A^T)^T A^T = A \cdot A^T \qquad \text{(bitno!!!)}$$

⇒ uvod u matrične jednadžbe:

PR: Odredi „X" tako da vrijedi:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} X = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

⇒ „X" je matrica tipa 3×2

$$X = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} c & d \\ a+e & b+f \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$c = 3$

$d = 5$

$a + e = 0 \Rightarrow e = -a$

$b + f = 1 \Rightarrow f = 1 - b$

$a, b \in R$

$$X = \begin{bmatrix} a & b \\ 3 & 5 \\ -a & 1-b \end{bmatrix}$$

DETERMINANTE

⇒ determinanta je vezana kvadratne matrice sa linearnim sustavima

$$\left.\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}\right\}(*) \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \tilde{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \quad \tilde{b} = \begin{bmatrix} e \\ f \end{bmatrix}$$

↳ matrični zapis (*):

$$A\tilde{X} = \tilde{b}$$

$$\begin{array}{l} ax + by = e \quad | \cdot d \quad | \cdot a \\ cx + dy = f \quad | \cdot b \quad | \cdot c \end{array}$$

$$adx + bdy = ed$$
$$bcx + bdy = fb$$

$$x = \frac{ed - bf}{ad - bc} \qquad y = \frac{af - ce}{ad - bc}$$

↳ $ad - bc \neq 0$

$$x = \frac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}, \qquad y = \frac{\begin{vmatrix} a & e \\ c & f \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \quad \det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

↳ CRAMEROVO PRAVILO

⇒ Cramerovo pravilo se može generalizirati na sve linearne sustave (kvadratne matrice)

$$\boxed{Ax = b}, \quad A \in \mathbb{R}^{n \times n}, \quad x \in \mathbb{R}^n, \quad b \in \mathbb{R}^n$$

DETERMINANTA: realan broj pridružen svakoj kvadratnoj matrici

⇒ DETERMINANTA MATRICE 2 × 2:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \quad \det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{12} \cdot a_{21}$$

⇒ DETERMINANTA MATRICE 3 × 3:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \qquad \det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$\det A = a_{11} \cdot \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \cdot \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

↳ ovako smo determinantu matrice 3 × 3 svel na determinantu matrice 2 × 2

↳ to napravimo tako da izaberemo red te član po član vidimo van i množimo sa determinantom matrice koju dobijemo ako ne gledamo redak i stupac člana koj smo izvukli van

⇒ DETERMINANTA MATRICE TIPA $n \times n$:

↳ neki pojmovi:

- MINORA: determinanta matrice koja se dobije brisanjem i-tog retka i j-tog stupca (oznaka M_{ij})

- ALGEBARSKI KOMPLEMENT: definira se ovako:

$$A_{ij} = (-1)^{i+j} \cdot M_{ij}$$

↳ napomena: svejedno je po kojem stupcu ili retku razvijemo determinantu, rezultat je uvijek isti

PR: Razvij determinantu po drugom retku!

$$\begin{vmatrix} 2 & -1 & 3 \\ 3 & 1 & 5 \\ 1 & 2 & -3 \end{vmatrix} = \sum_{j=1}^{3} a_{ij} A_{ij} \qquad \forall i \in \{1, 2, 3\}$$

$$= 3 \cdot A_{21} + 1 \cdot A_{22} + 5 \cdot A_{23}$$

⇒ općenit razvoj determinante matrice $n \times n$:

↳ LAPLACEOV RAZVOJ:

$$\det A = \sum_{j=1}^{n} a_{ij} A_{ij} \qquad ; \; i = 1, 2, \ldots, n$$ } razvoj po retcima

$$\det A = \sum_{i=1}^{n} a_{ij} A_{ij} \qquad ; \; j = 1, 2, \ldots, n$$ } razvoj po stupcima

→ napomena: računanje determinante je numerički vrlo zahtjevno po Laplaceovom razvoju (složenost $n!$)

↳ PR: ako imamo matricu 100×100, računalo od 10^{12} flops (floating point operations per second), trebalo bi nam $8 \cdot 10^{144}$ godina da dobijemo rezultat

(BIG BANG: prije $12.5 \cdot 10^{9}$ godina)

SVOJSTVA DETERMINANTI

$$A=\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 2 & 2 \\ 1 & 2 & -2 & 2 & 2 \\ 1 & 2 & 2 & -2 & 2 \\ 1 & 2 & 2 & 2 & -2 \end{bmatrix}$$

→ kasnije ćemo se vratiti na ovu matricu

⇒ svojstva izrađena za retke, ali je sve analogno za stupce

SVOJSTVO 1:

↳ ako A ima nul-redak, onda je $\det A = 0$

SVOJSTVO 2: (vrlo dobar)

↳ determinanta trokutaste matrice jednaka je umnošku elemenata na dijagonali

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & & a_{nn} \end{vmatrix} = a_{11}\cdot(-1)^{1+1}\cdot\begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & a_{nn} \end{vmatrix} = a_{11}\cdot a_{22}\cdot\ldots\cdot a_{nn}$$

SVOJSTVO 3: (pogledaj dobar)

↳ ako matrica ima dva jednaka retka, $\det A = 0$

SVOJSTVO 4: (dobar ne treba)

↳ $\det(A^T) = \det A$

SVOJSTVO 5: (bitno ▽▽▽) + (dokaz)

↳ determinanta se množi sa skalarom tako da se jedan njezin redak pomnoži s tim skalarom

$$\lambda \det A = \lambda \cdot \sum_{j=1}^{n} a_{ij} A_{ij} = \sum_{j=1}^{n} (\lambda a_{ij}) A_{ij}$$

↳ napomena: zapamti da vrijedi i za stupce

SVOJSTVO 6: (bitno ▽▽▽) + (dokaz)

↳ rastave li se svi elementi nekog retka matrice na zbroj dvaju elemenata, onda je determinanta jednaka zbroju odgovarajućih determinanti

PR:
$$\begin{vmatrix} x+2 & 3x-3 & -x+4 \\ 2 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} x & 3x & -x \\ 2 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix} + \begin{vmatrix} 2 & -3 & 4 \\ 2 & 0 & 1 \\ 4 & 3 & 1 \end{vmatrix}$$

⇒ dokaz: „stropimo" retke u vektor-retke

$$\begin{vmatrix} \vec{a}_1' + \vec{a}_1'' \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix} = \begin{vmatrix} \vec{a}_1' \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix} + \begin{vmatrix} \vec{a}_1'' \\ \vec{a}_2 \\ \vdots \\ \vec{a}_n \end{vmatrix}$$

↳ pokažimo ovo preko Laplace-ovog razvoja:

$$\det A = \sum_{j=1}^{n} (a_{1j}' + a_{1j}'') A_{1j} = \sum_{j=1}^{n} a_{1j}' A_{1j} + \sum_{j=1}^{n} a_{1j}'' A_{1j} =$$

$$= \det A' + \det A''$$

SVOJSTVO 7: (pogledaj dokaz)

↳ ako zamijenimo dva retka matrice, determinanta matrice mijenja predznak

svojstvo 3 ↓ svojstvo 6 ↓

$$0 = \begin{vmatrix} \vdots \\ \vec{a}_i + \vec{a}_j \\ \vdots \\ \vec{a}_i + \vec{a}_j \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_i + \vec{a}_j \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_i + \vec{a}_j \\ \vdots \end{vmatrix} = \underbrace{\begin{vmatrix} \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_i \\ \vdots \end{vmatrix}}_{0} + \begin{vmatrix} \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_j \\ \vdots \end{vmatrix} + \begin{vmatrix} \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_i \\ \vdots \end{vmatrix} + \underbrace{\begin{vmatrix} \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_j \\ \vdots \end{vmatrix}}_{0}$$

svojstvo 3 ↓

$$\Downarrow$$

$$\begin{vmatrix} \vdots \\ \vec{a}_i \\ \vdots \\ \vec{a}_j \\ \vdots \end{vmatrix} = - \begin{vmatrix} \vdots \\ \vec{a}_j \\ \vdots \\ \vec{a}_i \\ \vdots \end{vmatrix}$$

SVOJSTVO 8: (bitno ∇∇∇) + (dokaz)

↳ ako nekom retku matrice dodamo neki drugi redak pomnožen sa skalarom, vrijednost determinante se neće promijeniti

$$\begin{vmatrix} \vdots \\ \vec{a}_1 \\ \vdots \\ \lambda\vec{a}_1 + \vec{a}_2 \\ \vdots \end{vmatrix} = \lambda \underbrace{\begin{vmatrix} \vdots \\ \vec{a}_1 \\ \vdots \\ \vec{a}_1 \\ \vdots \end{vmatrix}}_{0} + \begin{vmatrix} \vdots \\ \vec{a}_1 \\ \vdots \\ \vec{a}_2 \\ \vdots \end{vmatrix} = \begin{vmatrix} \vdots \\ \vec{a}_1 \\ \vdots \\ \vec{a}_2 \\ \vdots \end{vmatrix}$$

svojstvo 3 ↓

↳ napomena: ovo svojstvo često koristimo za računanje determinante matrica reda $n \geq 4$

⇒ PR: Vratimo se na početni primjer i odredimo determinantu matrice.

$$\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -2 & 2 & 2 & 2 \\ 1 & 2 & -2 & 2 & 2 \\ 1 & 2 & 2 & -2 & 2 \\ 1 & 2 & 2 & 2 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 1 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 0 & 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & 1 & -3 \end{vmatrix} =$$

1. redak množimo sa (-1), dodajemo ostalima

$$= 1 \cdot (-1)^{1+1} \cdot \begin{vmatrix} -3 & 1 & 1 & 1 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 0 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 1 & 1 & 1 & -3 \end{vmatrix} = 0$$

razvoj po prvom stupcu

1. retku dodamo zbroj preostalih redaka

SVOJSTVO 9: Binet-Cauchyjev teorem

↳ bez dokaza zapamti:

$$\det(AB) = (\det A) \cdot (\det B)$$

⇒ napomena: ortogonalne matrice:

$$AA^T = A^T A = I$$

↳ te matrice imaju veliku primjenu, ali je za nas dovoljno da znamo što su

ZAD: Kolika iznos determinanta ortogonalne matrice?

$$1 = \det(AA^T) = (\det A) \cdot \underbrace{(\det A^T)}_{\det A} = (\det A)^2$$

$$\det A = \pm 1$$

RANG I INVERZ MATRICE

⇒ DEFINICIJA:

↳ matrica A' za koju vrijedi:

$$AA' = A'A = I$$

, naziva se INVERZNA MATRICA matrice A

↳ oznaka: A^{-1}

⇒ DEFINICIJA:

↳ za matricu $A \in \mathcal{M}_n$ kažemo da je regularna ukoliko postoji A^{-1}

⇒ napomena: inverzna matrica je jedinstvena (ukoliko postoji)

⇒ matrice koje nisu regularne, nazivamo SINGULARNE MATRICE

⇒ KRITERIJ ZA ODREĐIVANJE JE LI MATRICA A REGULARNA:

- A je regularna, tada $\exists A^{-1}$

$$AA^{-1} = A^{-1}A = I$$

$$1 = \det(AA^{-1}) = (\det A)(\det A^{-1})$$

↳ zaključak: $\boxed{\det A \neq 0}$

• $\det A \neq 0$, treba pokazati da postoji A^{-1} ?

$$A^{-1} = \frac{1}{\det A} \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}^T$$

↳ no, ovo nije toliko bitno (nemoj učiti izvod, pa ni formulu) jer kompleksnost raste brzinom $n!$

ZAPAMTI: $A \in \mathcal{M}_n$ je regularna $\Leftrightarrow \det A \neq 0$

⇒TEOREM: (pogledaj dokaz)

$A, B \in \mathcal{M}_n$ su regularne matrice

↳ tada je AB regularna matrica i vrijedi $(AB)^{-1} = B^{-1} A^{-1}$

↳DOKAZ:

1) $\det(AB) = \underbrace{\det A}_{\neq 0} \cdot \underbrace{\det B}_{\neq 0} \neq 0$

2) a) $(AB)(B^{-1}A^{-1}) = AB B^{-1} A^{-1} = A A^{-1} = I$

b) $(B^{-1}A^{-1})(AB) = B^{-1} A^{-1} A B = B^{-1} B = I$

ZAD: Neka je $A \in \mathcal{M}_n$ regularna matrica. Pokaži:

$$(A^T)^{-1} = (A^{-1})^T$$

$$\underbrace{AA^{-1}} = \underbrace{A^{-1}A} = I$$

$$AA^{-1} = I \;/^T \qquad A^{-1}A = I \;/^T$$

$$(A^{-1})^T A^T = I \qquad A^T (A^{-1})^T = I$$

$$\Rightarrow (A^T)^{-1} = (A^{-1})^T$$

⇒ što je, npr. $(A^{-1})^{-1}$?

$$AA^{-1} = I \;/^{-1} \qquad A^{-1}A = I \;/^{-1}$$

$$(A^{-1})^{-1} A^{-1} = I \qquad A^{-1} (A^{-1})^{-1} = I$$

usporedi s ovom formulom: $AA^{-1} = A^{-1}A = I$

↳ zaključak

$$(A^{-1})^{-1} = A$$

ZAD: Dokaži da je inverz regularne simetrične matrice također simetrična matrica ∇.

↳ treba pokazati: $(A^{-1})^T = A^{-1}$

DOKAŽI SAM ∇

UVOD U MATRIČNE JEDNADŽBE

$XA = B$ $\quad A, B \in \mathcal{M}_m$; A je regularna matrica

$XA = B \;/ \cdot A^{-1}$

$X = BA^{-1}$

ZAD: Riješi matričnu jednadžbu $(X^{-1}A)^{-1} = I + BX$?

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$$

↳ prvo provjeri jesu li matrice regularne kako bi uopće postoji i inverz te matrice !

$\det A \neq 0 \qquad \det B = 0$

↳ ne postoji B^{-1}

↳ sada, krenimo u rješavanje:

$$(X^{-1}A)^{-1} = I + BX$$

$$A^{-1}(X^{-1})^{-1} = I + BX$$

$$A^{-1}X = I + BX$$

$$A^{-1}X - BX = I$$

$$\underbrace{(A^{-1} - B)}_{C} X = I$$

$$C = A^{-1} - B$$

$$C = \frac{1}{1}\begin{bmatrix} 7 & -2 \\ -3 & 1 \end{bmatrix} - \begin{bmatrix} 2 & -2 \\ -3 & 3 \end{bmatrix}$$

$$C = \begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix} \Rightarrow \det C = 5 \cdot (-2) = -10 \neq 0$$

↳ „C" je regularna matrica

$$C^{-1} = -\frac{1}{10}\begin{bmatrix} -2 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

↳ vratimo se na jednadžbu:

$$C X = I \;/ \cdot C^{-1}$$

$$X = C^{-1} = \begin{bmatrix} \frac{1}{5} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

koristimo: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$A^{-1} = \frac{1}{\det A}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

ZAD: Izračunaj: $AX^{-1}B = BA$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

↳ vidimo računanjem determinant da su i A i B regularne matrice

$$A^{-1} \cdot / \; AX^{-1}B = BA \; / B^{-1}$$

$$X^{-1} = A^{-1}BAB^{-1} \; / {}^{-1}$$

$$X = (A^{-1}BAB^{-1})^{-1}$$

↳ pokažimo koliko je $(ABCD)^{-1}$!

⋮ Pogledaj u knjižici !

$$(A_1 A_2 \dots A_m)^{-1} = A_m^{-1} \cdot \dots \cdot A_2^{-1} A_1^{-1}$$

$$X = BA^{-1}B^{-1}A \longrightarrow \text{dalje je samo račun !}$$

ELEMENTARNE TRANSFORMACIJE

⇒ REDUCIRANI OBLIK MATRICE:

stožerni elementi

$$A = \begin{bmatrix} 0 & -1 & 3 & 0 \\ -2 & 5 & -4 & -5 \\ 2 & -1 & -8 & 5 \end{bmatrix} \Rightarrow A_R = \begin{bmatrix} 1 & 0 & -\frac{11}{2} & \frac{5}{2} \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

↳ iz reducirane matrice možemo zaključiti koliki je rang matrice

↳ RANG MATRICE: broj ne-nul redaka u reduciranom obliku matrice

↳ oznaka: $r(A)$ = „neki broj"

⇒ što je onda reducirani oblik matrice?

↳ oblik po pravilima:

- prvi ne-nulti element (STOŽER) je uvijek 1, a ostali u tom stupcu moraju biti 0
- svi reci koji sadrže samo nule, moraju biti ispod ostalih elemenata
- svaki naredni stožer mora biti desno od prethodnog

⇒ ELEMENTARNE TRANSFORMACIJE:

↳ rade samo nad retcima

1) zamjena dvaju redaka

2) množenje nekog retka sa skalarom različitim od nule

3) dodavanje nekog retka pomnoženog skalarom nekom drugom retku

PR: Transformirajmo matricu iz početnog primjera ∇:

1. KORAK: zamjena 1. i 3. retka:

$$\underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E_1} \cdot \underbrace{\begin{bmatrix} 0 & -1 & 3 & 0 \\ -2 & 5 & 4 & -5 \\ 2 & -1 & -8 & 5 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & -1 & -8 & 5 \\ -2 & 5 & -4 & -5 \\ 0 & -1 & 3 & 0 \end{bmatrix}}_{A_1}$$

E_1 je elementarna matrica

2. KORAK: dijeljenje 1. retka sa 2:

$$\underbrace{\begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2} \underbrace{\begin{bmatrix} 2 & -1 & -8 & 5 \\ -2 & 5 & -4 & -5 \\ 0 & -1 & 3 & 0 \end{bmatrix}}_{A_1} = \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & -4 & \frac{5}{2} \\ -2 & 5 & -4 & -5 \\ 0 & -1 & 3 & 0 \end{bmatrix}}_{A_2}$$

E_2 je elementarna matrica

3. KORAK: množenje 1. redka s 2 i dodavanje 2. redku:

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3} \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & -4 & \frac{5}{2} \\ -2 & 5 & -4 & -5 \\ 0 & -1 & 3 & 0 \end{bmatrix}}_{A_2} = \underbrace{\begin{bmatrix} 1 & -\frac{1}{2} & -4 & \frac{5}{2} \\ 0 & 4 & -12 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix}}_{A_3}$$

E_3 je elementarna matrica

4. KORAK: dijeljenje drugog retka s 4

⋮

5. KORAK: 2. redak množi s $\frac{1}{2}$ i dodaj ga 1. retku

⋮

6. KORAK: 2 redak množi s 1 dodaj 3. redku

⋮

↳ **UOČI:**

- elementarne matrice su regularne
- inverz elementarne matrice je opet elementarna matrica

⇒ DEFINICIJA:

↳ ako je matrica B dobivena iz matrice A nizom elementarnih transformacija, onda kažemo da su A, B **ekvivalentne matrice** (po retcima)

↳ PIŠEMO: $A \sim B$

⇒ ZAKLJUČAK: $A \sim A_R$

- ekvivalentne matrice imaju isti rang

⇒ stoga, reciprocirajmo:

$$A = \begin{bmatrix} 0 & -1 & 3 & 0 \\ -2 & 5 & -4 & -5 \\ 2 & -1 & -8 & 5 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & -8 & 5 \\ -2 & 5 & -4 & -5 \\ 0 & -1 & 3 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{2} & -4 & \frac{5}{2} \\ -2 & 5 & -4 & -5 \\ 0 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -\frac{1}{2} & -4 & \frac{5}{2} \\ 0 & 4 & -12 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & -\frac{1}{2} & -4 & \frac{5}{2} \\ 0 & 1 & -3 & 0 \\ 0 & -1 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -\frac{11}{2} & \frac{5}{2} \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = A_R$$

⇒ UOČI: $A \in \mathcal{M}_{m,n}$

$$r(A) \leq \min\{m, n\}$$

ZAD: U ovisnosti o $\lambda \in R$ odredite rang matrice A!

$$A = \begin{bmatrix} \lambda & 1 & 1 & 1 \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \\ 1 & 1 & 1 & \lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & \lambda \\ 1 & \lambda & 1 & 1 \\ 1 & 1 & \lambda & 1 \\ \lambda & 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & \lambda \\ 0 & \lambda-1 & 0 & 1-\lambda \\ 0 & 0 & \lambda-1 & 1-\lambda \\ 0 & 1-\lambda & 1-\lambda & 1-\lambda^2 \end{bmatrix} \begin{matrix} \\ \\ \\ / \cdot \frac{1}{1-\lambda} \end{matrix}$$

$\lambda \neq 1$

↳ 1. slučaj: $\lambda = 1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 1$

$\lambda = 1$

↳ 2. slučaj: $\lambda \neq 1$

$$\sim \begin{bmatrix} 1 & 1 & 1 & \lambda \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -1 & -1 \\ 0 & 1 & 1 & 1+\lambda \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & \lambda+1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & \lambda+2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & \lambda+2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & \lambda+3 \end{bmatrix} / \frac{1}{\lambda+3}$$

$\lambda \neq -3$

↳ 2.1 slučaj: $\lambda = -3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 3$

$\lambda = 3$

↳ 2. 2 slučaj: $\lambda \neq -3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & \lambda+2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad r(A) = 4, \quad \lambda \neq -3$$

$\Rightarrow$ rješenje: $\lambda = 1$, $r(A) = 1$

$\lambda = -3$, $r(A) = 3$

$\lambda \in R \setminus \{1, -3\}$, $r(A) = 4$

DODATNO O EKVIVALENTNIM MATRICAMA

$B = E_r \cdot \ldots \cdot E_2 \cdot E_1 \cdot A \quad \Rightarrow A \sim B$

$\Rightarrow$ ekvivalentne matrice imaju isti rang

$\Rightarrow A \in \mathcal{M}_n$ je regularna, $A \sim B$

↳ je li onda i B regularna?

DA!

↓ pokažimo

1. NAČIN: preko determinante $\rightsquigarrow \det B \neq 0$

2. NAČIN: $B^{-1} = (E_r \cdot \ldots \cdot E_1 A)^{-1} = \underbrace{A^{-1} E_1^{-1} \cdot \ldots \cdot E_r^{-1}}$

produkt regularnih matrica je regularna matrica

DEFINICIJA: $A \in \mathcal{M}_n$, kažemo da ima puni rang ako je $r(A) = n$

↳ moći: $A \in \mathcal{M}_n$ je ranga $n \Leftrightarrow A_R = I$

$\Rightarrow$ TEOREM: $A \in \mathcal{M}_n$ je regularna $\Leftrightarrow$ ima puni rang

RAČUNANJE INVERZNE MATRICE

$A \in \mathcal{M}_n$, $r(A) = n$

$A \sim I$ tj. $I = E_r \cdot \ldots \cdot E_2 E_1 A \quad / \cdot A^{-1}$

↳ odnosno: $A^{-1} = E_r E_{r-1} \cdot \ldots \cdot E_2 \cdot E_1$

nećemo svaku put računati elementarne matrice, pa je ovo više teorijskog karaktera

⇒ mi radimo ovako:

$$\left[\begin{array}{ccc|ccc} a_{11} & a_{12} \ldots & a_{1n} & 1 & 0 \ldots & 0 \\ a_{21} & a_{22} \ldots & a_{2n} & 0 & 1 \ldots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} \ldots & a_{nn} & 0 & 0 \ldots & 1 \end{array}\right] \qquad AX = I$$

$$\rightarrow [A \mid I]$$

↳ radimo elementarne transformacije:

$$[A \mid I] \sim [E_1 A \mid E_1] \sim [E_2 E_1 A \mid E_2 E_1] \sim \ldots$$

$$\sim \underbrace{[E_r \cdots E_1 A}_{A_R I} \mid \underbrace{E_r E_{r-1} \cdot \ldots \cdot E_2 E_1]}_{A^{-1}}$$

PRIMER

$$A = \begin{bmatrix} -3 & 1 & -1 \\ 1 & 0 & 1 \\ -2 & 2 & 3 \end{bmatrix}, \quad A^{-1} = ?$$

$$\left[\begin{array}{ccc|ccc} -3 & 1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ -3 & 1 & -1 & 1 & 0 & 0 \\ -2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 3 & 0 \\ 0 & 2 & 5 & 0 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -4 & 1 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 5 & -1 \\ 0 & 1 & 0 & 5 & 11 & -2 \\ 0 & 0 & 1 & -2 & -4 & 1 \end{array}\right]$$

A^{-1}

ZAD: Postoj li inverz matrice $a \cdot a^T + I$ gdje je $a = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$? Ako postoj, izračunajte ga!

$a \rightsquigarrow 4 \times 1 \qquad a \cdot a^T \rightsquigarrow 4 \times 4$

$$a \cdot a^T = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$a \cdot a^T + I = \begin{bmatrix} 2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

⋮

dalje sam!

BITNO!

ZAD: Rješi jednadžbu: $A^{-2}X + X = B$

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 0 & -1 & 1 \\ 1 & -2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad R_J = \frac{1}{5}\begin{bmatrix} 6 & -7 \\ -1 & 2 \\ 6 & -7 \end{bmatrix}$$

↳ napomena:

$$A^{-2} = A^{-1} \cdot A^{-1}$$

↳ važan detalj:

$$A^{-k} = (A^{-1})^k = (A^k)^{-1}$$

dokaz ↓

$$AA^{-1} = A^{-1}A = I$$

a) $AA^{-1} = I \;/^k$

$$\underbrace{AA^{-1} \cdot AA^{-1} \cdot \ldots \cdot AA^{-1}}_{k} = I$$

$$A^k (A^{-1})^k = I$$

b) $A^{-1}A = I \;/^k$

$$(A^{-1})^k A^k = I$$

$$(A^k)^{-1}$$

LINEARNA NEZAVISNOST VEKTORA I RANG MATRICE

$\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$ su vektor-stupci, R^m (V^m)

⇒ *linearna kombinacija vektora* $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$:

↳ to je vektor oblika:

$$\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \ldots + \lambda_k \vec{a}_k$$

$$\lambda_1, \ldots, \lambda_k \in R$$

↳ *skup svih linearnih kombinacija vektora* $\vec{a}_1, \ldots, \vec{a}_k$ *nazivamo prostorom razapetim vektorima* $\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k$

↳ oznaka $L(\vec{a}_1, \ldots, \vec{a}_k$

PR: $\vec{a}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\vec{a}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$;

$$L(\vec{a}_1, \vec{a}_2) = \{\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 : \lambda_1, \lambda_2 \in R\}$$

↳ UOČI: bilo koj vektor iz R^2 se može prikazat kao linearna kombinacija vektora $\vec{a}_1$ i $\vec{a}_2$

OKRENI

$\Rightarrow$ zaista, uzmimo $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in R^2$

$$\vec{X} = \lambda_1 \vec{a_1} + \lambda_2 \vec{a_2}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$X_1 = \lambda_1 + \lambda_2 \quad \Rightarrow \quad \lambda_2 = X_1 - X_2$$

$$X_2 = \lambda_1 \quad \Rightarrow \quad \lambda_1 = X_2'$$

BITNO!

ZAD: Odredi λ tako da vektor $\vec{b}$ bude linearna kombinacija vektora $\vec{a_1}$ i $\vec{a_2}$!

$$\vec{a_1} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} \quad \vec{a_2} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix} \quad \vec{b} = \begin{bmatrix} 7 \\ -2 \\ \lambda \end{bmatrix}$$

$$\vec{b} = \lambda_1 \vec{a_1} + \lambda_2 \vec{a_2}$$

$$\begin{bmatrix} 7 \\ -2 \\ \lambda \end{bmatrix} = \lambda_1 \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$$

$$7 = 2\lambda_1 + \lambda_2$$

$$-2 = \lambda_1 \cdot 3 + \lambda_2 \cdot 4$$

$$\lambda = 5\lambda_1 + 3\lambda_2$$

$$\vdots$$

$$\lambda_1 = 6$$

$$\lambda_2 = -5$$

$$\Downarrow$$

$$\lambda = 15$$

⇒ LINEARNA NEZAVISNOST:

• $\mathbb{R}^2 \rightarrow \vec{b}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \vec{b}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

↳ jedan se množenu napisat pomoću drugog pa su ti vektori linearno nezavisni

$$\lambda_1 \vec{b}_1 + \lambda_2 \vec{b}_2 = \vec{0}$$

$$\lambda_1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Downarrow$$

$$\lambda_1 = 0$$
$$\lambda_2 = 0$$

⇒ kažemo da su vektori $\vec{a}_1, \ldots, \vec{a}_k$ linearno nezavisni, ako iz jednakosti:

$$\lambda_1 \vec{a}_1 + \lambda_2 \vec{a}_2 + \ldots + \lambda_k \vec{a}_k = \vec{0}$$

... slijedi da su skalari λ_i moraju biti jednaki nula !

⇒ vektori su linearno ravisni ako nisu linearno ravisni

⇒ NAPOMENA:

$\vec{a_1}, \ldots, \vec{a_k}$ su linearno zavisni ako postoje skalari $\lambda_1, \ldots, \lambda_k$ takvi da barem jedan od njih nije jednak nul takvi da vrijedi:

$$\lambda_1 \vec{a_1} + \lambda_2 \vec{a_2} + \ldots + \lambda_3 \vec{a_3} = \vec{0}$$

↳ drugim riječima: barem jedan vektor se može prikazat kao linearna kombinacija preostalih

⇒ VEZA LINEARNE NEZAVISNOSTI I RANGA MATRICE:

PRS

$$A = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & -2 \\ 0 & 1 & -1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

↳ matrica je već u reduciranom obliku

$$r(A) = 3$$

↳ VEZA: rang matrice odgovara broju linearno nezavisnih ne-nul redaka!

$$\vec{a_1} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ -2 \end{bmatrix} \quad a_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} \quad a_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix}$$

OKRENI

$$\lambda_1 \vec{a_1} + \lambda_2 \vec{a_2} + \lambda_3 \vec{a_3} = \vec{0}$$

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 3 \\ -2 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 2 \\ 1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 4 \end{bmatrix} = \vec{0}$$

↳ izdvajamo jednadžbe:

$$\lambda_1 = 0$$

$$\lambda_2 = 0$$

$$2\lambda_1 - \lambda_2 = 0$$

$$\lambda_3 = 0$$

$$3\lambda_1 + 2\lambda_2 + 2\lambda_3 = 0$$

$$-2\lambda_1 + \lambda_2 + 4\lambda_3 = 0$$

↳ jedino rješenje:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0$$

↓

stoga, vektori su linearno nezavisni!

⇒ pogledajmo sada stupce radarne matrice!

↳ stupci:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$ su linearno nezavisni!

↳ svi ostali stupci mogu se zapisati kao linearna kombinacija ova tri stupca

VAŽNI REZULTATI: (ne dokazujemo)

- elementarne transformacije ne mijenjaju broj linearno nezavisnih redaka
- rang matrice jednak je broju linearno nezavisnih redaka matrice (DEFINICIJA RANGA)
- broj linearno nezavisnih redaka matrice jednak je broju njezinih linearno nezavisnih stupaca - u ovo se uklapa: $r(A) = r(A^T)$

ZAD: Za koje „a" su vektori $\begin{bmatrix}1\\2\\3\\1\end{bmatrix}$, $\begin{bmatrix}2\\-1\\3\\1\end{bmatrix}$, $\begin{bmatrix}4\\3\\9\\a\end{bmatrix}$ linearno neovisni?

$$\lambda_1\begin{bmatrix}1\\2\\3\\1\end{bmatrix}+\lambda_2\begin{bmatrix}2\\-1\\3\\1\end{bmatrix}+\lambda_3\begin{bmatrix}4\\3\\9\\a\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$

↳ ispišimo jednadžbe:

$$\left.\begin{aligned}\lambda_1+2\lambda_2+4\lambda_3&=0\\2\lambda_1-\lambda_2+3\lambda_3&=0\\3\lambda_1+3\lambda_2+9\lambda_3&=0\\\lambda_1+\lambda_2+a\lambda_3&=0\end{aligned}\right\}$$ pravokutni linearni sustav (broj nepoznanica nije jednak broju jednadžbi)

$$X=\begin{bmatrix}\lambda_1\\\lambda_2\\\lambda_3\end{bmatrix} \quad ; \quad AX=0$$

$$A=\begin{bmatrix}1&2&4\\2&-1&3\\3&3&9\\1&1&a\end{bmatrix}$$

↳ homogeni linearni sustav (nula s desne strane) (uvijek ima rješenje, al nam je bitno je li jedino rješenje nul-vektor ili ne)

↳ rješimo ovaj zadatak pomoću ranga jer još nismo upoznati s linearnim sustavima

$$A=\begin{bmatrix}1&2&4\\2&-1&3\\3&3&9\\1&1&a\end{bmatrix}\sim\begin{bmatrix}1&2&4\\0&-5&-5\\0&-3&-3\\0&-1&a-4\end{bmatrix}\sim\begin{bmatrix}1&2&4\\0&1&1\\0&1&1\\0&-1&a-4\end{bmatrix}\sim$$

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & a-3 \end{bmatrix} \dots$$

↳ za $a = 3$:

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 2 < 3$

→ vektori su linearno zavisni

↳ za $a \neq 3$:

$$\sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$r(A) = 3$

→ vektori su linearno nezavisni

LINEARNI SUSTAVI

⇒ općeniti linearni sustav:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mm}x_m &= b_m \end{aligned} \quad .$$

MATRIČNI ZAPIS ↓

$$Ax = b \qquad A \in \mathbb{R}^{m \times m},\ x \in \mathbb{R}^m,\ b \in \mathbb{R}^m$$

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & & a_{mm} \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

↳ što znači riješit linearni sustav?

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_m \begin{bmatrix} a_{1m} \\ a_{2m} \\ \vdots \\ a_{mm} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

⇒ riješit linearni sustav znači vektor „b" zapisat kao linearnu kombinaciju vektora stupaca matrice A

GAUSSOVA METODA ELIMINACIJE

⇒ IDEJA: elementarnim transformacijama nad retcima matrice sustava, zadani sustav svodimo na njemu ekvivalentni sustav (u smislu da ti sustavi imaju isti skup rješenja) iz kojega možemo jednostavno očitati rješenja

PR: Imamo „kvadratni sustav":

$$\begin{aligned} X_1 + 2X_2 + 3X_3 &= 5 \\ 2X_1 - X_2 - X_3 &= 1 \\ X_1 + 3X_2 + 4X_3 &= 6 \end{aligned}$$

↳ računamo elementarne transformacije:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & -1 & -1 & 1 \\ 1 & 3 & 4 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & -5 & -7 & -9 \\ 0 & 1 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 7 & 9 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \Rightarrow X_1 = 1 \\ \Rightarrow X_2 = -1 \\ \Rightarrow X_3 = 2 \end{array}$$

HOMOGENI SUSTAVI

⇒ *homogen sustav:*

↳ *sustav koj na desnoj strani ima nul-vektor*

$$A x = \vec{0}$$

$$a_{11} x_1 + a_{12} x_2 + \dots + a_{1m} x_m = 0$$
$$\vdots$$
$$a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mm} x_m = 0$$

↳ *homogen sustav uvijek ima rješenje* $(x = 0)$, *ali je pitanje da l je to jedino rješenje.*

TEOREM: *(bitno - vidi dokaz)*

↳ *neka je* $A \in \mathcal{M}_m$ *kvadratna matrica*

↳ *jednadžba* $Ax = 0$ *ima jedinstveno rješenje* $x = 0$ *ako i samo ako je A regularna*

$$x_1 \vec{a}_1 + x_2 \vec{a}_2 + \dots + x_m \vec{a}_m = \vec{0} \quad (*)$$

$\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$ *su stupci matrice A*

↳ dobar „udesno":

PRETPOSTAVKA: $Ax = 0$ ima jedinstveno rješenje $x = 0$

↳ treba pokazati da je A regularna:

- Iz $(*)$ i pretpostavke slijedi da su stupci matrice A linearno nezavisni, pa je $r(A) = n$, dakle (A) je regularna

↳ dobar „ulijevo":

PRETPOSTAVKA: A je regularna matrica

↳ treba pokazati da je $x = 0$ jedinstveno rješenje sustava $Ax = 0$:

- Ako je A regularna, onda je $r(A) = n$, pa su stupci linearno nezavisni, pa iz $(*)$ direktno slijedi tvrdnja //

PR:

$$X_1 + X_2 = 0$$
$$X_1 + X_3 = 0$$
$$X_2 - X_3 = 0$$

$X_1 + X_3 = 0 \Rightarrow X_1 = -X_3$

$X_2 - X_3 = 0 \Rightarrow X_2 = X_3$

slobodna varijabla

vezane varijable

uvijek je 0, pa ne treba pisat

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

$X_1 \; X_2 \; X_3$

$r(A) = 2 < 3$, pa je

A singularna matrica

PR: $X_1 + 2X_2 + X_3 + 2X_4 + 3X_5 = 0$

$2X_1 + 4X_2 + 3X_3 + 3X_4 + 7X_5 = 0$

$2X_1 + 4X_2 + X_3 + 5X_4 + 5X_5 = 0$

↳ izračunamo reducirani oblik:

$$A_r = \begin{matrix} X_1 & X_2 & X_3 & X_4 & X_5 \end{matrix}$$
$$\begin{bmatrix} 1 & 2 & 0 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

stožeri

X_1, X_3 ⇒ vezane varijable (nalaze se u stupcima gdje su stožeri)

X_2, X_4, X_5 ⇒ slobodne varijable

↳ dimenzija rješenja:

$$\boxed{d = n - r}$$

d ⇒ dimenzija rješenja

n ⇒ broj nepoznanica

r ⇒ rang matrice

$d = 5 - 2 = 3$

↳ UOČI: dimenzija rješenja je broj slobodnih varijabli

$X_1 + 2X_2 + 3X_4 + 2X_5 = 0$

$X_3 - X_4 + X_5 = 0$

$X_1 = -2X_2 - 3X_4 - 2X_5 = -2\alpha - 3\beta - 2\gamma$

$X_3 = X_4 - X_5 = \beta - \gamma$

$$x_2 = \alpha \;,\quad x_4 = \beta \;,\quad x_5 = \gamma$$

$$\alpha,\ \beta,\ \gamma \in \mathbb{R}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2\alpha - 3\beta - 2\gamma \\ \alpha \\ \beta - \gamma \\ \beta \\ \gamma \end{bmatrix} = \alpha \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -3 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \gamma \begin{bmatrix} -2 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

NEHOMOGENI SUSTAVI

⇒ sustav oblika:

$$Ax = b \qquad b \neq 0$$

$$A \in R^{m \times n}, \; x \in R^{n}, \; b \in R^{m}$$

⇒ nas zanima egzistencija rješenja (ima li rješenja?) i zanima nas jedinstvenost rješenja

PR: $2X + 3Y = 4$

$4X + 6Y = 7$

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}; \quad \tilde{X} = \begin{bmatrix} x \\ y \end{bmatrix}; \quad b = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & 3 & 4 \\ 4 & 6 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 3 & 4 \\ 0 & 0 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & \frac{3}{2} & 2 \\ 0 & 0 & -1 \end{array}\right]$$

$$X + \frac{3}{2}Y = 2$$

$$0 \cdot X + 0 \cdot Y = -1 \quad \Rightarrow \text{NEMOGUĆE!}$$

↳ ovaj sustav nema rješenja!

⇒ UOČI: u ovom proširom primjeru, rang matrice A je različit od ranga proširene matrice A

↳ ovo će nam biti kriterij za određivanje egzistencije rješenja

↳ zašto? Pa vidimo da nam matrica onda izgleda:

$$\left[\begin{array}{ccc|c} \vdots & & & \vdots \\ 0 & 0 \cdots & 0 & a \end{array}\right] \qquad a \neq 0$$

TEOREM: Kronecker - Cappellyev teorem

(ne treba dokazat)

↳ sustav $Ax = b$ ima rješenje ako i samo ako je $r(A) = r(A_p)$

A_p ⇒ proširena matrica matrice A

TEOREM:

↳ opće rješenje sustava $Ax = b$ je oblika:

$$X = X_h + X_p$$

gdje je X_h opće rješenje pripadnog homogenog sustava ($Ax = 0$), a X_p jedno (partikularno) rješenje nehomogenog sustava

OKRENI (vodoravno)

DOKAZ:

- neka je: $Ax = b$,

  x_p je proizvoljno rješenje nehomogenog sustava //

$$\left.\begin{aligned} Ax &= b \\ Ax_p &= b \end{aligned}\right\}$$

$$Ax - Ax_p = \vec{0}$$

$$A(\underbrace{x - x_p}_{x_h}) = \vec{0} \quad \Rightarrow \quad x - x_p = x_h$$

$$x = x_h + x_p$$

↳ sada dokazujemo obratno:

- neka je: $x = x_h + x_p$

$$A(x_h + x_p) = Ax_p + Ax_h = b$$

ZAD: Neka su „u" i „v" dva rješenja linearnog sustava $Ax = b$, $A \in M_{mn}$. Nađite sve linearne kombinacije vektora „u" i „v" koje su također rješenje danog sustava!

$$Au = b \quad , \quad Av = b$$

$$b = A(\alpha u + \beta v)$$ → to su linearne kombinacije $\alpha, \beta \in R$

$$b = \alpha A u + \beta A v = \alpha b + \beta b = (\alpha + \beta) b$$

$\Downarrow$ slijed

$$\alpha + \beta = 1 \Rightarrow \beta = \alpha - 1$$

↳ dakle, sve linearne kombinacije izgledaju ovako:

$$\alpha u + (1 - \alpha) v \quad ; \quad \alpha \in R$$

⇒ UOČI: $Ax = b$

↳ sustav nema rješenja il ima točno jedno rješenje il ima beskonačno mnogo rješenja

⇒ ALGORITAM za rješavanje linearnih sustava:

↳ sustav je oblika:

$$Ax = b, \quad A \in \mathcal{M}_{m,n}, \; x \in V^n, \; b \in V^m$$

- $[A \mid b]$
- Gaussova metoda $[A_R \mid b']$
- ako je: 1) $r(A) < r(A_p)$

↳ sustav nema rješenja

2) $r(A) = r(A_p)$

↳ rješivo sustav

$$d = n - r$$

↳ za $r = n \Rightarrow d = 0$ imamo jedinstveno rješenje

↳ za $r < n \Rightarrow d > 0$ imamo beskonačno mnogo rješenja

ZAD: $X + 2Y + 3Z = 3$

$-2X \quad + Z = -2$

$X + 2Y - Z = 3$

$-X + 2Y + 12Z = 1$

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ -2 & 0 & 1 & -2 \\ 1 & 2 & -1 & 3 \\ -1 & 2 & 12 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 4 & 7 & 4 \\ 0 & 0 & -4 & 0 \\ 0 & 4 & 15 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 4 & 7 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 8 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 4 & 7 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow r(A) = 3$$

$$r(A_p) = 3$$

↳ sustav ima rješenje

- $d = n - r = 3 - 3 = 0$

↳ sustav ima jedinstveno rješenje ▽!

↳ Sada, možemo dalje raditi transformacije ili jednostavno povratnim supstitucijama (od dolje prema gore) isčitamo rješenja

↳ krenimo dalje transformirati:

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \begin{array}{c} X \quad Y \quad Z \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

↳ $X = 1$

$Y = 1$

$Z = 0$

TEOREM:

↳ neka je $A \in M_{nn}$ (kvadratna matrica)

↳ tada, za po volji odabran $b \in V^n$ sustav $Ax = b$ ima točno jedno rješenje onda i samo onda ako je A regularna matrica

DOKAZ:

- A je regularna

  ↳ $r(A) = n = r(A_p)$ $\quad \left(r \leq \min(\underset{\substack{\uparrow \\ n}}{m}, \underset{\substack{\uparrow \\ n+1}}{n}) \right)$

  ↓

  zaključujemo da sustav ima rješenje, ali još treba pokazat jedinstvenost:

$$x = \underset{\substack{\uparrow \\ 0}}{x_h} + x_p$$

jer je A regularna

- $Ax = b$ ima jedinstveno rješenje $\forall b \in R^n$

  ↳ uzmimo, npr. $b = 0$ (nul-vektor)

  ↓

  tada sustav $Ax = 0$ ima jedinstveno rješenje $(x = 0)$, pa odatle slijedi da je A regularna matrica

PR: $A \in \mathcal{M}_3$

$$b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in R^3$$

$$\begin{aligned} x_1 + x_2 + 3x_3 &= b_1 \\ x_2 + 2x_3 &= b_2 \\ 2x_1 + 3x_2 + 8x_3 &= b_3 \end{aligned}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 2 & 3 & 8 & b_3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 1 & 2 & b_3 - 2b_1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 0 & 0 & b_3 - 2b_1 - b_2 \end{array}\right]$$

Imamo dva slučaja!

1. slučaj:

- $b_3 - 2b_1 - b_2 = 0$

$r(A) = 2$; $r(A_p) = 2$

$d = 3 - 2 = 1$

↳ sustav ima beskonačno rješenja

2. slučaj:

- $b_3 - 2b_1 - b_2 \neq 0$

$r(A) = 2$; $r(A_p) = 3$

↳ sustav nema rješenja

↳ UOČI: polazna matrica A je bila singularna matrica

ZAD: Za koje vrijednosti $\lambda \in \mathbb{R}$ zadan sustav ima:
a) jedno rješenje
b) beskonačno rješenja (odredi rješenja te ih prikaži u vektorskom obliku)
c) nema rješenja

$$\begin{aligned} X_1 + 3X_2 + 3X_3 &= 1 \\ -X_1 + \lambda X_2 - 2X_3 - 3X_4 &= 2 \\ -X_1 - 3X_2 + \lambda X_3 - 2X_4 &= 1 \\ 2X_1 + 6X_2 + 7X_3 + \lambda X_4 &= 4 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ -1 & \lambda & -2 & -3 & 2 \\ -1 & -3 & \lambda & -2 & 1 \\ 2 & 6 & 7 & \lambda & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & 1 & -3 & 3 \\ 0 & 0 & \lambda+3 & -2 & 2 \\ 0 & 0 & 1 & \lambda & 2 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & 1 & -3 & 3 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & \lambda+3 & -2 & 2 \end{array}\right] \begin{array}{l} \\ \\ \\ \downarrow (-\lambda-3) \end{array} \sim \left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & 1 & -3 & 3 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & 0 & -(\lambda+1)(\lambda+2) & -2(\lambda+2) \end{array}\right] \begin{array}{l} \\ \\ \\ \cdot(-1) \end{array} \sim (*)$$

$$\lambda \cdot (-\lambda-3) - 2 = -\lambda^2 - 3\lambda - 2 = -(\lambda+1)(\lambda+2)$$

$$(*) \sim \left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & 1 & -3 & 3 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & 0 & (\lambda+1)(\lambda+2) & 2(\lambda+2) \end{array}\right]$$

1. SLUČAJ: $\lambda = -1$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & 2 & 1 & -3 & 3 \\ 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 2 \end{array}\right]$$

$$r(A) \neq r(A_p)$$

↳ sustav nema rješenja

2. SLUČAJ: $\lambda = -2$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & 1 & 1 & -3 & 3 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & 6 & -5 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 9 & -8 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$x_4 = t \in \mathbb{R}$

$$x_1 + 9x_4 = -8$$
$$x_2 - x_4 = 1$$
$$x_3 - 2x_4 = 2$$

$$x_1 = -8 - 9t$$
$$x_2 = 1 + t$$
$$x_3 = 2 + 2t$$

$$x = \begin{bmatrix} -8 - 9t \\ 1 + t \\ 2 + 2t \\ t \end{bmatrix}$$

$$x = \begin{bmatrix} -8 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 1 \\ 2 \\ 1 \end{bmatrix}$$

↳ vratimo se na matricu i pogledajmo slučaj

$\lambda \neq -1$, $\lambda \neq -2$!

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & 1 & -3 & 3 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & 0 & (\lambda+1)(\lambda+2) & 2(\lambda+2) \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & -3\lambda & 1 \\ 0 & \lambda+3 & 0 & (\lambda+3) & 3 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & 0 & (\lambda+1)(\lambda+2) & 2(\lambda+2) \end{array}\right]$$

→ PAZI !

3. SLUČAJ: $\lambda = -3$

↳ sustav nema rješenja

4. SLUČAJ: $\lambda \in \mathbb{R} \setminus \{-1, -2, -3\}$

$$\left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & \lambda+3 & 1 & -3 & 3 \\ 0 & 0 & 1 & \lambda & 2 \\ 0 & 0 & 0 & (\lambda+1)(\lambda+2) & 2(\lambda+2) \end{array}\right] \begin{array}{l} / \cdot \frac{1}{\lambda+3} \\ \\ \\ / \cdot \frac{1}{(\lambda+1)(\lambda+2)} \end{array} \sim$$

$$\sim \left[\begin{array}{cccc|c} 1 & 3 & 3 & 0 & 1 \\ 0 & 1 & \frac{1}{\lambda+3} & \frac{-3}{\lambda+3} & \frac{3}{\lambda+3} \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & \frac{2}{\lambda+1} \end{array}\right]$$

↳ sustav ima jedinstveno rješenje

SVOJSTVENE VRIJEDNOSTI

⇒ „poseban" vektori za neku matricu su vektori kod kojih se djelovanjem matrice na taj vektor „ništa smjer", odnosno vrijedi:

$$A\upsilon = \lambda\upsilon \quad , \quad (\lambda \text{ je skalar})$$

DEFINICIJA: $A \in \mathcal{M}_n$

- vektor $\upsilon \neq 0$ zovemo SVOJSTVENI (VLASTITI) vektor matrice A, ako postoji skalar λ tako da vrijedi:

$$\boxed{A \cdot \upsilon = \lambda \cdot \upsilon} \qquad \upsilon \neq 0$$

„λ" je svojstvena vrijednost matrice A

PR: $A = I \in \mathbb{R}^{n \times n}$

$$I\upsilon = \upsilon = 1 \cdot \upsilon$$

svojstvena vrijednost

$\upsilon \in \mathbb{R}^n$

Kako nalazimo svojstvene vektore i svojstvene vrijednosti proizvolje matrice $A \in \mathcal{M}_n$?

$$Av = \lambda v \quad ; \quad v \neq 0$$

$$(\lambda \cdot I - A)\, v = 0$$

$$\det(\lambda I - A) = 0$$

$\Rightarrow$ „v" je svojstveni vektor matrice A ako i samo ako ovaj homogen sustav ima rješenje $v \neq 0$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & a_{12} & \cdots & -a_{1n} \\ -a_{21} & \lambda - a_{22} & \cdots & -a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \lambda - a_{nn} \end{vmatrix}$$

PR: $A \in \mathcal{M}_2$

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{vmatrix} = (\lambda - a_{11})(\lambda - a_{22}) - a_{12} a_{21} =$$

$$= \lambda^2 - \lambda a_{22} - \lambda a_{11} - a_{12} a_{21} + a_{11} a_{22} =$$

$$= \lambda^2 - \lambda(a_{11} + a_{22}) - a_{12} a_{21} + a_{11} + a_{22}$$

↳ vidimo da sada treba rješiti ovu kvadratnu jednadžbu:

$$\lambda^2 - \lambda(a_{11} + a_{22}) + a_{11}a_{22} - a_{12}a_{21} = 0$$

↳ polinom 2. stupnja u varijabli λ

$k(\lambda)$

⇒ OPĆENITO:

$$k(\lambda) = \det(\lambda I - A) = \underbrace{\lambda^n - \sigma_1 \lambda^{n-1} - \dots - \sigma_{n-1}\lambda - \sigma_n}$$

karakteristični polinom matrice A

$$\sigma_1, \dots, \sigma_n \in R$$

⇒ svojstvene vrijednosti matrice A su nultočke karakterističnog polinoma. Ima ih točno „n", a mogu biti i kompleksne, no takve ne radimo u matematici 1 ▽ ▽ ▽

↳ napomena: „n" nultočaka ima ako brojimo s njihovu kratnost

PR: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda \end{vmatrix} = \lambda^2 + 1$$

ima kompleksne svojstvene vrijednosti!

⇒ NAPOMENA:

↳ isto je prem l:

$$(A - \lambda I)\, v = 0$$

„il"

$$(\lambda I - A)\, v = 0$$

ZAD: Odredi svojstvene vrijednosti i svojstvene vektore matrice:

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$Av = \lambda v \quad ; \quad v \neq 0$$

$$(\lambda I - A)v = 0 \quad , \quad v \neq 0$$

$$0 = \det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & -1 \\ -1 & \lambda & -1 \\ -1 & -1 & \lambda \end{vmatrix} = \ldots =$$

$$= \lambda(\lambda^2 - 1) + (-\lambda - 1) - (\lambda + 1) =$$

$$= \lambda(\lambda - 1)(\lambda + 1) - 2(\lambda + 1) =$$

$$= (\lambda + 1)\left(\lambda(\lambda - 1) - 2\right) =$$

$$= (\lambda + 1)^2 \cdot (\lambda + 1) \quad \Rightarrow \lambda_1 = \lambda_2 = -1$$
$$\lambda_3 = 2$$

↳ prva svojstvena vrijednost: $\lambda_1 = \lambda_2 = -1$

$$Av = (-1)v \quad ; \quad v \neq 0$$

$$(A + I)v = 0 \quad ; \quad v \neq 0$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow v_1 + v_2 + v_3 = 0$$

OKRENI

$$v_1 = -v_2 - v_3$$

$$v_2 = \alpha \quad ; \quad v_3 = \beta \qquad \alpha, \beta \in \mathbb{R}$$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -\alpha - \beta \\ \alpha \\ \beta \end{bmatrix} = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$\alpha, \beta \in \mathbb{R}$

$v \neq 0$ (ili svako $\alpha^2 + \beta^2 \neq 0$)

↳ druga svojstvena vrijednost: $\lambda_3 = 2$

$$Av = 2v \quad , \quad v \neq 0$$

$$(2I - A)v = 0 \quad , \quad v \neq 0$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix} \sim \dots \sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix} \sim$$

$$\sim \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{aligned} v_1 - v_3 &= 0 \\ v_2 - v_3 &= 0 \end{aligned}$$

$v_3 = t \in \mathbb{R}$

$$v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad t \in \mathbb{R} \setminus \{0\}$$